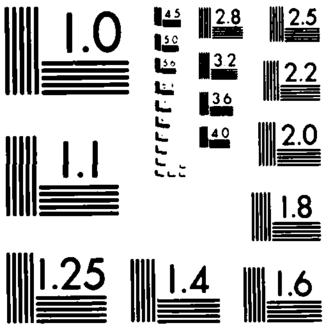
AD-A131 838 ACCURACY ESTIMATES OF GRAVITY POTENTIAL DIFFERENCES BETWEEN WESTERN EUROP..(U) OHIO STATE UNIV COLUMBUS DEPT OF GEODETIC SCIENCE AND SURVEYI.. D P HAJELA 1/1

UNCLASSIFIED FEB 83 OSU/DGSS-345 AFGL-TR-83-0132 F/G 8/5 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

AFGL-TR-83-0132

ACCURACY ESTIMATES OF GRAVITY POTENTIAL DIFFERENCES BETWEEN WESTERN EUROPE AND UNITED STATES THROUGH LAGEOS SATELLITE LASER RANGING NETWORK

D.P. Hajela

The Ohio State University
Research Foundation
Columbus, Ohio 43212

February 1983

Scientific Report No. 3

Approved for public release; distribution unlimited

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

83 08 23 104

CONTRACTOR REPORTS

This report has been reviewed by the ESD Public Affairs Office (PA) and is releasable to the National Technical Information Service (NTIS).

This technical report has been reviewed and is approved for publication.

Christopher Jekeli

CHRISTOPHER JEKELI
Contract Manager

Thomas P. Rooney

THOMAS P. ROONEY
Chief, Geodesy & Gravity Branch

FOR THE COMMANDER

Donald H. Eckhardt

DONALD H. ECKHARDT
Director
Terrestrial Sciences Division

Qualified requestors may obtain additional copies from the Defense Technical Information Center. All others should apply to the National Technical Information Service.

If your address has changed, or if you wish to be removed from the mailing list, or if the addressee is no longer employed by your organization, please notify AFGL/DAA, Hanscom AFB, MA 01731. This will assist us in maintaining a current mailing list.

Do not return copies of this report unless contractual obligations or notices on a specific document requires that it be returned.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFGL-TR-83-0132	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ACCURACY ESTIMATES OF GRAVITY POTENTIAL DIFFERENCES BETWEEN WESTERN EUROPE AND UNITED STATES THROUGH LAGEOS SATELLITE LASER RANGING NETWORK		5. TYPE OF REPORT & PERIOD COVERED Scientific Report No. 3 6. PERFORMING ORG. REPORT NUMBER OSU/DGSS-345
7. AUTHOR(s) D.P. Hajela		8. CONTRACT OR GRANT NUMBER(s) F19628-82-K- 0022
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Ohio State University Research Foundation 1958 Neil Avenue, Columbus, Ohio 43210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2309G1BC
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01730 Monitor/Christopher Jekeli/LWG		12. REPORT DATE February 1983 13. NUMBER OF PAGES 75
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

World Vertical Network, Gravity Potential Differences, Lageos SLR Network

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The presently attainable accuracy of connecting the vertical datums of regions, which are separated by oceans, has been investigated. The vertical datum connections are established through geocentric positions of laser tracking stations by estimating the geopotential at these stations, and also utilizing the geopotential differences between stations in each region obtained through leveling. The Western Europe-USA vertical datum connection has been examined in detail. Accuracy estimates of a few other datum connections are also given.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Four stations in Western Europe and 14 stations in USA were considered in the Lageos satellite laser ranging (SLR) network (January 1982). The currently available accuracy estimates (December 1981) of the potential coefficients describing the earth's gravity field to degree 180 were used, along with gravity anomalies in a small area around each SLR station.

Various anomaly spacing, data cap size, anomaly accuracy were tried to determine the effect on the accuracy of the vertical datum connection. It was found that anomaly spacing need not be more dense than 10'. The results below are quoted for anomaly accuracy of 2 mgals (1σ), but there was only slight deterioration in the results even if the anomaly accuracy was as poor as 10 mgals. The important finding was that anomalies in a data cap as small as 30' spherical radius would be adequate for establishing the vertical datum connection.

Different accuracy estimates for the SLR station positions, and any displacement of the SLR coordinate system origin from the geocenter, were considered. For conservative estimates of radial position accuracy of 10 cm (1σ) and lack of geocentricity of the SLR coordinate system of 30 cm (1σ), the standard deviation of Europe-US vertical datum connection was presently estimated as 58 kgal.cm. This was based on a conservative accuracy estimate of leveling for determining geopotential differences over intra-continental distances of a few 10^3 km. Even if the leveling accuracy was reduced by one-half, the vertical datum connection accuracy only became 59 kgal.cm. The corresponding values for the vertical connection, if anomalies were used in a data cap of 1° radius, were 50 and 52 kgal.cm.

Foreword

This report was prepared by Dhaneshwar P. Hajela, Associate Professor, Department of Geodetic Science and Surveying, The Ohio State University, under Air Force Contract No. F19628-82-K-00022,The Ohio State University Research Foundation Project No. 714274. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Dr. Christopher Jekeli, Scientific Program Officer.

Acknowledgements

I am indebted to Dr. Richard H. Rapp for his advice and support of this study. The procedure of establishing the vertical datum connection follows the ideas originally formulated and programmed by Dr. Oscar Colombo. Thanks are due to my friend, Dr. Demos Christodoulidis, for his discussion of the coordinates of the Lageos SLR network. The careful typing of this report by Miss Laura Brumfield is sincerely appreciated.

Table of Contents

List of Figures

List of Tables

1. INTRODUCTION

It is well recognized that the mean sea level, as a surface of reference, varies rapidly 'in time and space', e.g. see Lennon (1974). The variation of mean sea level from the geoid, or the 'quasi-stationary' non-tidal sea-surface topography, e.g. see Mather (1978), is not very well known but is of the order of 1 meter r.m.s. (root mean square). The vertical geodetic networks in different continents, which are respectively referenced to mean sea level determinations at coastal tide gages, therefore refer to different equipotential surfaces. The problems of unification of vertical datums have been briefly reviewed by Rapp (1980).

Colombo (1980a) proposed that a 'World Vertical Network' may be defined by 'a set of estimated potential differences among benchmarks situated in various continents' (ibid., p.2). These potential differences were postulated to be independent of the precise definition of the geoid, or the precise knowledge of the zero harmonic of the earth's gravity potential. The principle of Colombo's method is more fully described in Section 2, but in outline, he considered the potential difference between two benchmarks, say BMA and BMB, each in a different continent, by computing the normal and disturbing potential at each location, subject to the uncertainties in the knowledge of the earth's gravity field and station (benchmark) locations. The estimated potential difference, $\Delta W(BMA,BMB)$, could be made more precise through redundant determinations by computing the potential differences between some other pairs of benchmarks, P_i and Q_j in the respective continents; and using these later differences to estimate $\Delta W(BMA,BMB)$ by considering potential differences $\Delta W(BMA,P_i)$ and $\Delta W(BMB,Q_j)$ obtained through gravity and leveling observations.

Colombo considered the normal gravitational potential described by a spherical harmonic expansion to degree and order 20, with standard deviations of the potential coefficients ('imperfect' model) as determined by Rapp (1978), and also with zero standard deviation ('perfect' model). The disturbing potential was computed from gravity anomalies approximately uniformly distributed at a spacing of about 0.4° around each benchmark in a 'cap' of spherical radius 5° (10° was also considered in some cases).

With four benchmarks at hypothetical locations in Australia and five benchmarks in North America, with standard deviation of relative geocentric position as 0.15 m and the standard deviation of potential differences $\Delta W(BMA,P_i)$, $\Delta W(BMB,Q_j)$ as $0.1\sqrt{\ell}$ kgal.m, where ℓ is the length of leveled connection in 10^3 km, Colombo (1980a, p.33) estimated the sta d d d v [illegible] of the Australia - North America Vertical datum connection as about 0.3 kgal.m for the 'imperfect' model, and about 0.2 kgal.m for the 'perfect' model, i.e. the low degree potential coefficients to degree and order 20 being known perfectly without any error. Colombo (198 extended the results to consider a 0.3 m r.m.s. unknown shift of tl origin of the (benchmark) coordinate system from geocenter, when the ·timated standard deviation of the vertical datum connection rose to abo 4 and 0.3 kgal.m respectively for the imperfect and perfect gravitation odel. A slightly lower value was obtained for hypothetically located benchmarks for the North and South America vertical datum connection.

The precise coordinates are now available of the Lageos Satellite Laser Ranging (SLR) network (Smith et al., 1982) in support of the NASA Geodynamics program (NASA, 1981, p.6; NASA, 1982, p.7). Improved estimates are also now available of the potential coefficients in the spherical harmonic expansion of the earth's gravitational field (Rapp, 1981) to degree and order 180. The present study applies the ideas of Colombo (1980a, 1980b) to the currently available data, which is described in Section 3, to determine the current accuracy estimates of the vertical datum connections. The West Europe-United States vertical datum connection is examined in detail in Section 6, as a greater number of SLR stations are located in these areas. The current accuracy estimates are also determined for some other vertical datum connections in Section 7.

The present study first investigates in Section 4 the optimum requirement of the density and extent of gravity anomalies in the cap around each benchmark for estimating the disturbing potential at the cap center. Any reduction in the spherical radius of the cap from 5° makes the gravity anomaly data acquisition and reduction more feasible. This reduction in the cap radius is first examined in terms of the accuracy of the estimated

disturbing potential. Further reduction in cap radius is examined in Section 6 in terms of the accuracy of the vertical datum connection, as a larger number of cap pairs may be available for a reduced cap radius. Finally, an optimal minimum size of cap radius is recommended for a very modest requirement of gravity anomaly data while still ensuring reasonable accuracy estimates.

The hypothetical benchmark locations in Colombo's studies (1980a, 1980b) were placed away from coastal areas so that gravity anomalies may not be required over marine areas, as such anomalies would be of lower precision as compared to those obtainable over land areas. However, as several stations of the SLR network are located in coastal areas, the effect of lower precision of gravity anomalies is examined in Section 4.5.

The accuracy of potential difference estimates $\Delta W(BMA,P_i)$ and $\Delta W(BMB,Q_j)$ obtained through leveling could be of some concern over intra-continental distances, e.g., see Kumar and Soler (1981); Lachapelle and Gareau (1980, Section 4). The effect of leveling errors on the vertical datum connection are examined in Sections 6 and 7 of this report.

The effects of the unknown shift of the origin of SLR coordinate system from the geocenter,and the random position errors of the SLR stations, are also examined in Sections 6 and 7. The accuracy of the normal gravitational field is considered in Section 4.4.

2. MATHEMATICAL MODEL FOR THE VERTICAL DATUM CONNECTION

We summarize here the proposal of Colombo (1980a, 1980b) for establishing the vertical datum connection between two continents or regions A and B by estimating the potential difference $\Delta W(BMA,BMB)$ between two benchmarks BM A and BM B in the two regions. We assume that there are several benchmarks or stations P_i and Q_j respectively in the two regions, and that the potential differences $\Delta W_\ell(BMA,P_i)$ and $\Delta W_\ell(BMB,Q_j)$ have been obtained through leveling (and hence the subscript ℓ). Note that P_i and Q_j respectively include BMs. A and B, when obviously $\Delta W_\ell(BMA,BMA)$ and $\Delta W_\ell(BMB,BMB)$ are zero.

2.1 Accuracy Estimate of the Vertical Datum Connection

Denoting the gravity potential W as a sum of the normal gravitational potential U, the rotational potential Φ, and the anomalous potential T; and also introducing errors by the notation δ, we have

$$\begin{aligned}\Delta W(BMA,BMB) = {} & U(P_i) + T(P_i) + \Phi(P_i) + \Delta W_\ell(BMA,P_i) \\ & - [U(Q_j) + T(Q_j) + \Phi(Q_j) + \Delta W_\ell(BMB,Q_j)] + v_{ij}\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}v_{ij} = {} & \delta U(P_i) - \delta U(Q_j) + \delta T(P_i) - \delta T(Q_j) + \delta\Delta W_\ell(BMA,P_i) \\ & - \delta\Delta W_\ell(BMB,Q_j),\end{aligned} \tag{2.2}$$

and we are generally following Colombo's (ibid.) notation for ease of reference. The errors in position of SLR stations propagate to errors in U and Φ, but as the magnitude of errors $\delta\Phi$ is negligibly small ($<$ 0.3%) as compared to δU, the terms $\delta\Phi$ have been omitted in (2.2).

It is clear, by a simple consideration, that the maximum number, N_e, of linearly independent equations (2.1) between paris of benchmarks P_i, Q_j for establishing the vertical datum connection, is one less than the total number of benchmarks or SLR stations considered in the two regions. The actual choice of pairs is practically determined, though arbitrarily, for the simplicity of design (see Section 6.1).

Equation (2.1) may be written in matrix form as:

$$\underline{a} \ \Delta W(BMA,BMB) = \underline{p} + \underline{v} \tag{2.3}$$

where $\underline{a}$, $\underline{p}$, $\underline{v}$ are all vectors of dimension N_e. $\underline{a}$ is the design vector with all components equal to 1, $\Delta W(BMA,BMB)$ is the unknown potential difference for the vertical datum connection in the overdetermined system (2.3); $\underline{p}$ is the vector of 'observed' potential differences, and $\underline{v}$ is the vector of residuals.

The accuracy estimate, $\sigma \Delta W(BMA,BMB)$, of the vertical datum connection is obtained from:

$$\sigma \Delta W(BMA,BMB) = (\underline{a}^T V^{-1} \underline{a})^{-1/2} = [\sum_k^{N_e} \sum_l^{N_e} (V^{-1})_{kl}]^{-1/2} \tag{2.4}$$

the last form of the equation being a consequence of all components of $\underline{a}$ being 1. In (2.4), V is the variance-covariance matrix of the 'observed' potential differences:

$$V = V(\varepsilon\Delta U) + V(\hat{\varepsilon}\Delta T) + V(\varepsilon\Delta\Delta W_\ell) \tag{2.5}$$

where the right hand side terms are the variance-covariance matrices of the errors in $U(P_i) - U(Q_j)$, $\hat{T}(P_i) - \hat{T}(Q_j)$, $\Delta W_\ell(BMA,P_i) - \Delta W_\ell(BMB,Q_j)$ respectively. The computation of $V(\varepsilon\Delta U)$ and $V(\varepsilon\Delta\Delta W_\ell)$ is described in Sections 2.2 and 2.3, while $V(\hat{\varepsilon}\Delta T)$ is considered in Section 5 after considering the accuracy estimates of the predicted disturbing potential in Section 4.1. The notation $\hat{T}(P_i)$ or $\hat{T}(Q_j)$ indicates that 'modified' gravity anomalies (Section 4.2) have been used for predicting T. It is also clear from subsequent discussion that the errors $\varepsilon\Delta U$, $\hat{\varepsilon}\Delta T$, $\varepsilon\Delta\Delta W_\ell$ are primarily due to different parameters. The correlations between these errors have therefore been neglected in (2.5).

2.2 Variance Covariance Matrix of Normal Gravitational Potential Differences

The normal gravitational potential U at a station P with geocentric coordinates (r, ϕ', λ) may be denoted by:

$$U(r,\phi',\lambda) = \frac{GM}{r}[1+\sum_{n=2}^{N}(a/r)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi')(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)] \quad (2.5)$$

where GM is the gravitational constant times the mass of the earth; N is the highest degree of the spherical harmonic expansion; a here is the equatorial radius; $\bar{P}_{nm}$, $\bar{C}_{nm}$, $\bar{S}_{nm}$ are respectively the fully normalized Legendre's functions and the potential coefficients of degree n and order m.

The errors εU in (2.6) are due to errors in station coordinates, in the value of GM , and in the potential coefficients, i.e. εC_{nm}, εS_{nm}. We will consider the effect of errors εC_{nm}, εS_{nm} entirely while evaluating the error εT in Section 4.1; and consider their effect on εU here as zero. The effect of $\varepsilon GM/r$ will be almost zero on the potential differences as such errors cancel in the differencing. And, as already mentioned, the errors $\varepsilon\phi'$, $\varepsilon\lambda$ for the SLR stations will have negligibly small effect on εU , as compared to the effect of εr on εU. We may thus presently consider only the effect of errors εr , i.e.,

$$\varepsilon U(P_i) \simeq \frac{\partial U}{\partial r} \varepsilon r(P_i) \simeq -\bar{\gamma}\varepsilon r(P_i) \quad (2.7)$$

where it is sufficient to consider an average value $\bar{\gamma}$ of normal gravity on the ellipsoid. This approximation is reasonable as the latitudinal spread of the SLR network is not large (see Table 3.1). For simplicity, we further assume for this study that the standard deviation of radial position errors, $\sigma\varepsilon r$, has the same value for all stations of the SLR network. For actual vertical datum connections, it is not necessary to make these simplifying assumptions.

It is also reasonable to assume no correlation for εr in the determination of the positions of SLR stations, and we thus have a diagonal matrix for $V(\varepsilon\Delta U)$ with each diagonal term $\sigma^2\varepsilon\Delta U$ given by:

$$\sigma^2\varepsilon\Delta U = 2\bar{\gamma}^2\sigma^2\varepsilon r \quad (2.8)$$

We now need to also consider the additional contribution to $V(\varepsilon\Delta U)$ of the unknown shift of the SLR coordinate system from the geocenter. The error $\varepsilon r(P_i)$ caused by the shift (ΔX, ΔY, ΔZ) of the origin of the coordinate system is given by the projection of the shift on the radius vector $r(P_i)$:

$$\varepsilon r(P_i) = \frac{X_i \Delta X + Y_i \Delta Y + Z_i \Delta Z}{r(P_i)} \tag{2.9}$$

where (X_i, Y_i, Z_i) are the coordinates of P_i.

Denoting the coordinates of Q_j by (X_j, Y_j, Z_j), and the spherical distance between P_i and Q_j by $\psi(P_i,Q_j)$; the covariances in $V(\varepsilon\Delta U)$ in addition to (2.8) are obtained (see Colombo, 1980b, p.99, equation (21)) by using (2.7) and (2.9):

$$\bar{\gamma}^2 \operatorname{cov}(\varepsilon r(P_i), \varepsilon r(Q_j)) = \bar{\gamma}^2 \frac{X_i X_j + Y_i Y_j + Z_i Z_j}{r(p_i)\, r(Q_j)} \sigma^2\Delta$$

$$= \bar{\gamma}^2 \frac{\sigma^2 s}{3} \cos(\psi(P_i,Q_j)) \tag{2.10}$$

where the simplifying assumptions have been made that variances of shifts $(\sigma^2\Delta X, \sigma^2\Delta Y, \sigma^2\Delta Z)$ along coordinate axes are not correlated with each other, and are all equal to some constant $\sigma^2\Delta$:

$$\sigma^2\Delta X = \sigma^2\Delta Y = \sigma^2\Delta Z = \sigma^2\Delta = \sigma^2 s/3 \tag{2.11}$$

and σs is the standard deviation of the shift of the coordinate system origin. A range of values may be tried for σs , say from 0 to 50 cm.

2.3 Variance Covariance Matrix of Leveled Potential Differences

The standard deviation, σ, of the first order, class I double-run leveling, based on tolerances (Federal Geodetic Control Committee, 1980, pp.24 and 28) between forward and backward measurements, may be quoted as $0.5\sqrt{\ell}$ mm, where ℓ is the leveled distance in kilometers (Vanicek et al., 1980, p. 507). However, this may not be realistic for long lines of leveling observed over a period of several months. For the determination of ΔW_ℓ over intra-continental distances between SLR stations, a more conservative estimate (Personal Communication from C.T. Whalen, NGS to R.H. Rapp, June 1981) would be $\sigma = 2.10^{-}\sqrt{\ell}$ kgal.m. Now, if ℓ is expressed in 10^3 km, $\sigma = .06\sqrt{\ell}$ kgal.m, which is comparable to $0.1\sqrt{\ell}$ kgal.m used by Colombo (1980a,b). To get an insight into the effect of leveling errors on the vertical datum connection, a range of values will be tried for $\sigma \equiv \sigma\varepsilon\Delta W_\ell$ in Sections 6 and 7:

$$\sigma\varepsilon\Delta W_\ell = k(0.1\sqrt{\ell}) \text{ kgal.m}, \ \ell \text{ in } 10^3 \text{ km}, \ k = 0,1,2,4 \tag{2.12}$$

If ΔW_ℓ is obtained from BMA (or BMB) to P_i (or Q_j) by separate routes, the variances of $\sigma^2\varepsilon\Delta W_\ell$ would be uncorrelated. The leveling errors covariance matrix in one region would have off-diagonal elements if the level route proceeds to a station through a preceding station. For example, if in region B, there are five stations Q_j (including BMB, which is numbered as 1), and level routes are independent from BMB to each station except to station 5, which proceeds through station 4, then $V(\varepsilon\Delta W_\ell)_B$ may be written in the form of following matrix:

$$V(\varepsilon\Delta W_\ell)_B = .01\,k^2 \begin{bmatrix} (1,1) & & & & \\ & (1,2) & & & \\ & & (1,3) & & \\ & & & (1,4) & (1,4) \\ & & & (1,4) & (1,4)+(4,5) \end{bmatrix} \tag{2.13}$$

with other elements zero. The term (1,4), for example, is the distance between stations 1 and 4 in 10^3 km; and similarly, other non-zero terms in (2.13). The form of $V(\varepsilon\Delta W_\ell)$ is discussed further in Section 6.1 (also see Table 6.1).

The elements of $V(\varepsilon\Delta\Delta W_\ell)$ in (2.5) for leveling errors in the potential difference $\Delta W_\ell(BMA,P_i) - \Delta W_\ell(BMB,Q_j)$ would be formed by summation of corresponding elements in $V(\varepsilon\Delta W_\ell)_A$ and $V(\varepsilon\Delta W_\ell)_B$, according to the selected pairing of stations in the two regions in the N_e equations represented by (2.1), assuming no covariances between leveling errors in region A with those in region B.

3. STATIONS AND GRAVITATIONAL MODEL

The stations used for computing the accuracy estimates for the vertical datum connection were from the solution SL4 (Smith et al., 1982), and are described in Section 3.1. The gravitation model is described in Section 3.2. Potential differences ΔW_ℓ, through leveling and gravity observations in a region, were not needed (see Section 2.3 for leveling error covariances) for the vertical datum connection accuracy estimates. But estimates for ΔW_ℓ should be readily available when actual computations are made. A very wide range of $\sigma\varepsilon\Delta W_\ell$ in (2.12) will be considered for the present accuracy studies in Sections 6 and 7.

It is assumed for this study that point gravity anomalies will be available in a uniform pattern at an approximate spacing of 10' (18-20 km) around the SLR stations in a cap. These uniformly patterned anomalies may be predicted from existing gravity anomalies in the area. (See Section 4.1. Additional gravity observations may be required in some areas). 'Modified' anomalies (Colombo, 1980b, Sec. 3.3) are envisaged for predicting $T(P_i)$ or $T(Q_j)$, which will be discussed in Section 4.2. The choice of an optimally minimum spherical radius of the cap for the vertical datum connection would be discussed in Section 6.3.

3.1 Lageos Satellite Laser Ranging Stations

A precise set of coordinates are available in the SL4 system (Smith et al., 1982) from Lageos laser ranging data for May 1976 till August 1981. A slightly revised list of coordinates was supplied on punched cards by D. Christodoulidis (Personal communication, June 1982). Counting only as one station, when different nearby sites were occupied at different times, there were 29 stations in the solution. Stations 7896 (Pasadena) and 7110 (Monument Peak) were very close, about 1° and 1/2° respectively, to stations 7115 (Goldstone) and 7062 (San Diego). After omitting 7896 and 7110, the remaining 27 stations have been plotted in Figure 3.1. There are 14 stations in conterminous United States, 4 in Europe, 2 each in Australia and South America, 2 each in the Carribean and Pacific Islands, and 1 in Hawaii.

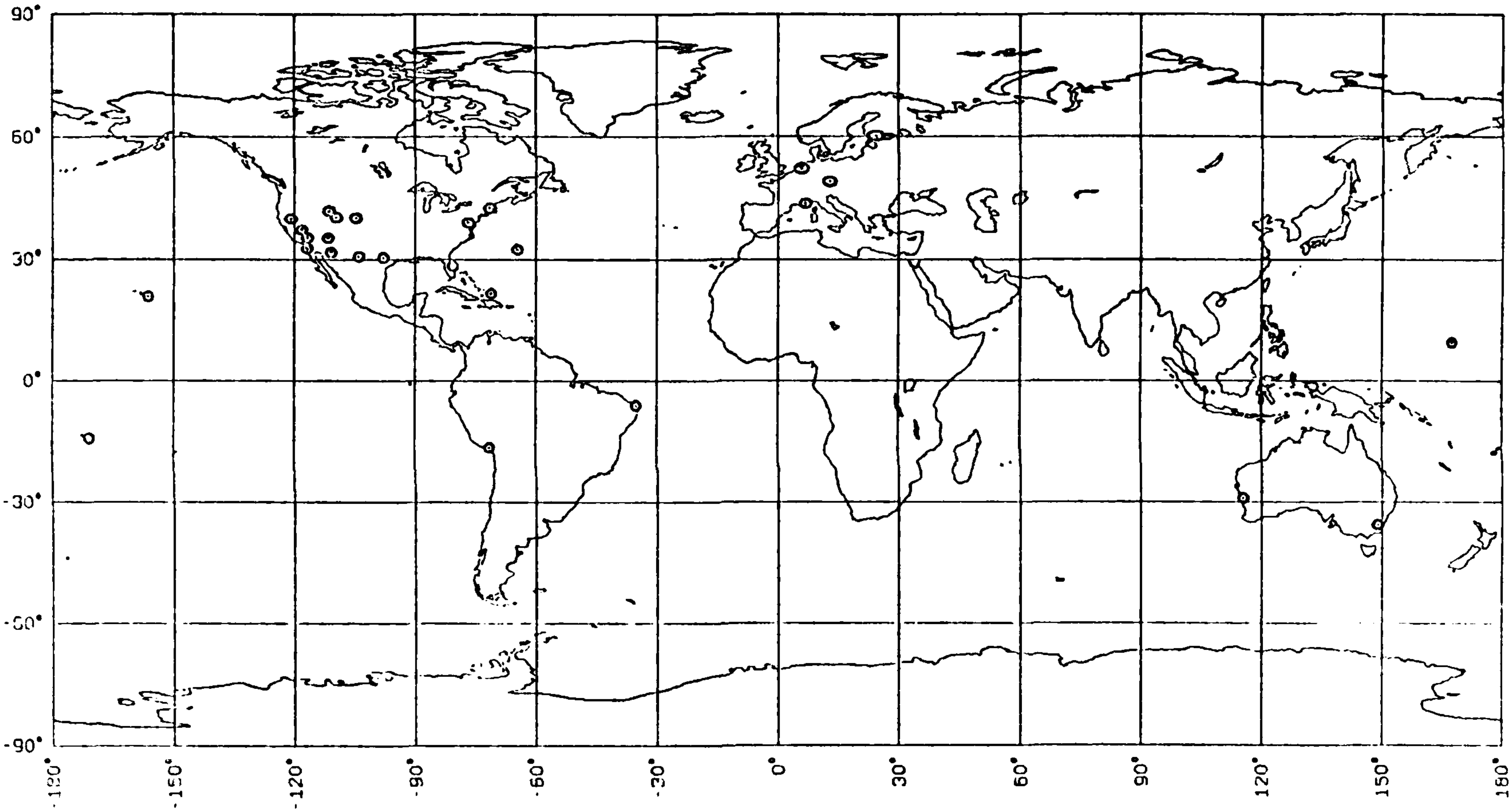


Figure 3.1: Lageos Satellite Laser Ranging Stations in the SL4 System. (Smith et al., 1982). Stations 7110 and 7896 have not been plotted.

Table 3.1 SLR Stations Used for Europe - USA Vertical Datum Connection

Seq.	Station		Latitude [2]			Longitude [2]			Height [2]	L/X [1]	Location
	Name	No.	°	'	"	°	'	"	m		
1	WETZEL	7834	49	08	41.7787	12	52	40.9657	654.1492	L	Wetzell, Germany
2	GRASSE	7835	43	45	16.8/55	6	55	15.8526	1315.6276	L	Grasse, France
3	KOOLAS	7833	52	10	42.2545	5	48	35.1147	86.3254	L	Kootwijk, Holland
4	FINLAS	7805	60	13	2.2465	24	23	40.2314	71.4034	L	Metsahovi, Finland
5	PLTTVX	7112	40	10	58.0009	255	16	26.3396	1494.483	X	Platteville, Colorado
6	VERNLX	7892	40	19	36.6456	250	25	44.8838	1582.945	X	Vernal, Utah
7	BERLKX	7082	41	56	00.8957	248	34	45.5485	1955.7989	X	Bear Lake, Utah
8	QUINCX	7051	39	58	24.5673	239	03	37.5587	1052.8915	X	Quincy, California
9	OVR79X	7114	37	13	57.2092	241	42	22.2217	1170.9681	X	Owens Valley, Calif.
10	GLD79X	7115	35	14	53.8975	243	12	28.9554	1031.5438	X	Goldstone, California
11	SANDIX	7062	32	36	02.6536	243	09	32.7974	981.5779	X	San Diego, California
12	FTDAVX	7086	30	40	37.3008	255	59	2.4888	1954.3317	X	McDonald Obsy., Texas
13	HOPLAS	7921	31	41	03.2180	249	07	18.8530	2345.4446	L	Mt. Hopkins, Arizona
14	FLAGSX	7891	35	12	52.3333	248	21	55.6195	2137.297	X	Flagstaff, Arizona
15	TLRBCX	7890	30	18	55.7924	262	08	03.8911	250.2249	X	Austin, Texas
16	STALSX	7063	39	01	13.3581	283	10	19.8002	12.216	X	Greenbelt, Maryland
17	HAYSTX	7091	42	37	21.6825	288	30	44.3454	84.9723	X	Haystack Obsy., Mass
18	RAMLSX	7069	28	13	40.6423	279	23	39.3278	-30.754	X	Patrick AFB, Florida

[1] L = Laser Tracker Optical Center; X = Survey Pad. [2] The Geodetic Coordinates are for the SL4 Solution (Smith et al., 1982), with Ellipsoid Parameters: $a = 6{,}378{,}144.11$ m, $1/f = 298.255$

Because the largest number of SLR stations are in Europe and U.S., the vertical datum connection between these two regions has been examined in detail in Section 6. The coordinates of these 18 stations have been listed in Table 3.1. The coordinates refer to the Laser tracker optical center at the four sites in Europe and the SAO site at Mt. Hopkins, Arizona. The coordinates at the remaining 11 sites refer to the survery pad.

3.2 Model of the Earth's Gravitational Field

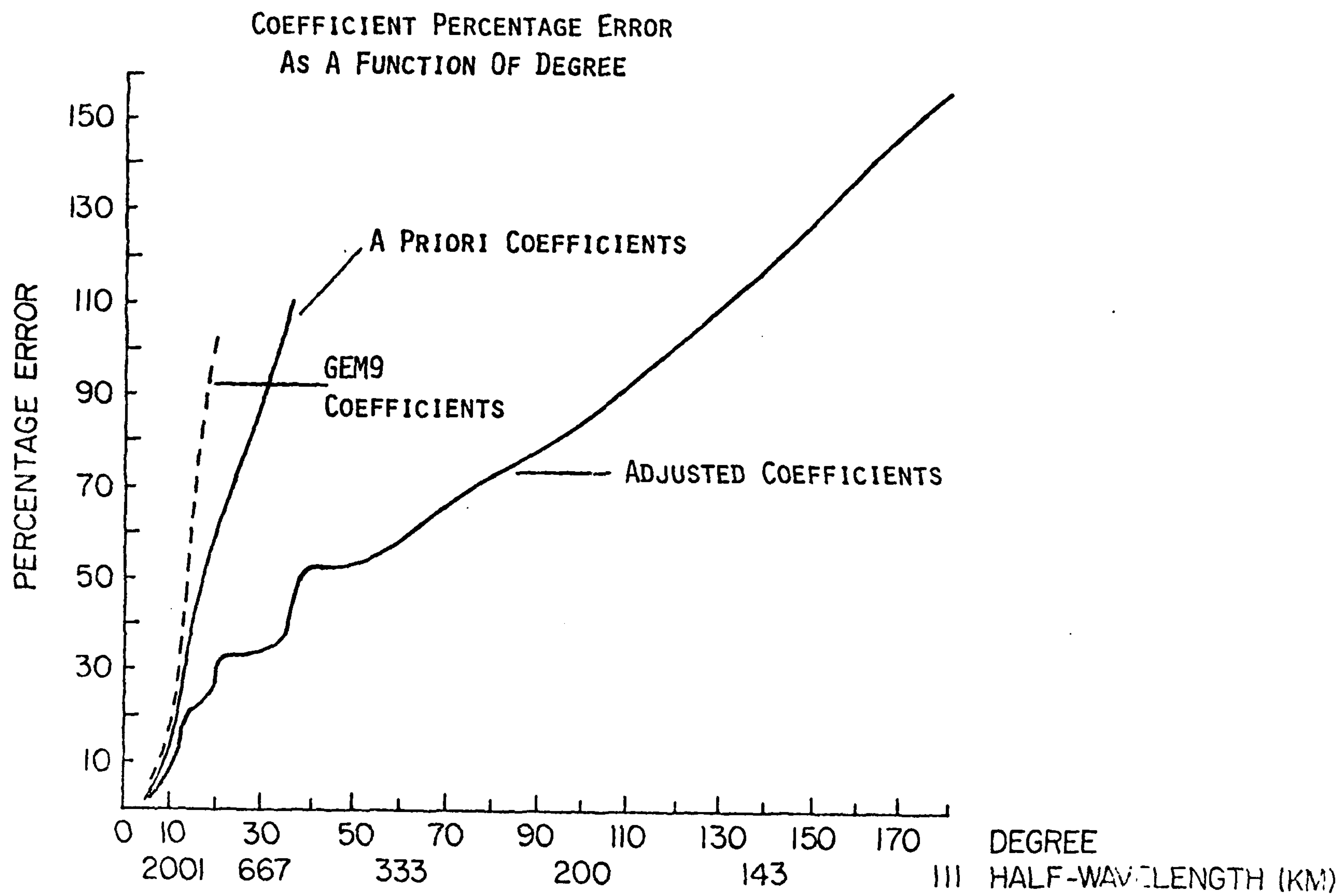
Rapp (1981) computed the potential coefficients in the spherical harmonic expansion (2.6) of the earth's gravitational field to degree and order 180. The expansion was in fact carried out to degree 300, but there was some concern (ibid., p. 24) about the sharp discontinuity at degree 180 in the quadrature weights for computing the potential coefficients. The data sets utililized for developing this field were an a-priori set of potential coefficients to degree 36 based on a number of the latest available solutions including a substantial number of resonance terms, a $1°\times 1°$ anomaly field derived from the Seasat data set, and the latest available $1°\times 1°$ terrestrial field. In principle, the three data sets were used to provide a global set of adjusted $1°\times 1°$ anomalies, from which the final set of potential coefficients were obtained by the quadrature formulas.

The standard deviation ($e\bar{C}_{nm}$, $e\bar{S}_{nm}$) of the potential coefficients was estimated considering the data noise, i.e. the r.m.s. standard deviation of the global anomaly set, which was taken as 20 mgals; and the sampling error due to the finite size of $1°\times 1°$ blocks.

The effect of data noise is several times larger than the effect of sampling error on the standard deviation of the potential coefficients (Rapp, 1981, p. 29). Figure 11 (ibid., p. 33) shows the coefficient percentage error, $\%E_n$, as a function of degree n. This has been reproduced as Figure 3.2 and shows that $\%E_n$ is about 60% at degree 60 and rises to about 150% at degree 180.

$$\%E_n = \Big[\sum_{m=0}^{n} (e^2\bar{C}_{nm}+e^2\bar{S}_{nm})/(2n+1)\Big]^{\frac{1}{2}}\Big/\Big[\sum_{m=0}^{n} (\bar{C}^2_{nm}+\bar{S}^2_{nm})/(2n+1)\Big]^{\frac{1}{2}}\cdot 100 \quad (3.1)$$

Figure 3.2: December 1981 Potential Coefficients Field (Reproduced from Rapp, 1981, p. 33, Fig. 11) See Table 3.2.



The rise in $\%E_n$ reflects the fact that coefficients decay faster than the estimated standard deviation. (Also see Section 4.4, Figure 4.4). There is also some evidence that the standard deviations may be slightly pessimistic (Lachapelle and Rapp, 1982, p. 4). The r.m.s. coefficient and the r.m.s. accuracy by degree i.e. the denominator and numerator on the right hand side of (3.1) have been listed in Table 3.2 for all degrees for $N \leq 30$, and every tenth degree till N = 180.

Table 3.2

RMS Variation of Coefficient, σ_n, and RMS Accuracy, e_n by Degree N
December 1981 Potential Coefficients Field (Rapp, 1981)

N	$\sigma_n \cdot 10^8$	$e_n \cdot 10^8$	N	$\sigma_n \cdot 10^8$	$e_n \cdot 10^8$	N	$\sigma_n \cdot 10^8$	$e_n \cdot 10^8$
			21	1.22	.34	90	.14	.11
2	125.7	.11	22	1.40	.34			
3	112.2	.30	23	1.05	.32	100	.12	.10
4	50.5	.31	24	.92	.32			
5	35.0	.59	25	.96	.31	110	.104	.094
6	24.9	.48	26	.73	.30			
7	19.7	.71	27	.67	.27	120	.089	.087
8	11.7	.64	28	.75	.27			
9	9.7	.81	29	.65	.26	130	.074	.080
10	7.7	.65	30	.70	.24			
11	5.6	.76	40	.48	.25	140	.065	.075
12	3.3	.57						
13	4.4	.60	50	.39	.20	150	.053	.070
14	2.5	.54						
15	2.3	.49	60	.29	.17	160	.048	.066
16	2.6	.50						
17	2.0	.44	70	.20	.15	170	.043	.063
18	1.8	.42						
19	1.6	.41	80	.17	.13	180	.038	.060
20	1.2	.37						

$$\sigma_n = \left[\sum_{m=0}^{n} (\bar{C}_{nm}^2 + \bar{S}_{nm}^2)/(2n+1)\right]^{\frac{1}{2}} ; \quad e_n = \left[\sum_{m=0}^{n} (e^2\bar{C}_{nm} + e^2\bar{S}_{nm})/(2n+1)\right]^{\frac{1}{2}}$$

The potential coefficients in Rapp (1981) will be used in this study to describe the earth's normal gravitational field. The coefficients set to degree 180, or a lower degree subset, will be termed as December 1981 potential coefficients field when the standard deviations as given in ibid. are used. It will be termed as 'perfect to N=10, 20, 30' if the standard deviations of the coefficients to degree 10, 20, 30 are set to zero. This corresponds to the characterization by Colombo (1980 a,b) of the 'imperfect' and 'perfect' model of the normal gravitational field.

By using (2.6), the anomalous potential $T(P)$ at a station P, with geocentric coordinates (r,ϕ',λ), may be expressed by:

$$T(P) = \frac{GM}{r}\{\sum_{n=2}^{N}(a/r)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi')\,[\varepsilon\bar{C}_{nm}\cos m\lambda + \varepsilon\bar{S}_{nm}\sin m\lambda]$$

$$+ \sum_{n=N+1}^{\infty}(a/r)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi')\,[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda]\} \qquad (3.2)$$

where we have now considered the errors $\varepsilon\bar{C}_{nm}$, $\varepsilon\bar{S}_{nm}$ in the normal gravitational field to degree N in defining $T(P)$.

We may similarly write an expression for the gravity anomaly $\Delta g(P)$ using the spherical approximation relation (Heiskanen and Moritz, p. 89):

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}\cdot T \qquad (3.3)$$

The covariances involving T and Δg at points P and P' are then given by (Colombo, 1980a, p. 13):

$$\mathrm{cov}(T(P),\Delta g(P')) = \frac{G^2M^2}{rr'^2}\{\sum_{n=2}^{N}\left(\frac{a^2}{rr'}\right)^n (2n+1)(n-1)\,\varepsilon_n^2\, P_n(\cos\psi_{PP'})$$

$$+ \sum_{n=N+1}^{\infty}\left(\frac{a^2}{rr'}\right)^n (2n+1)(n-1)\,\sigma_n^2\, P_n(\cos\psi_{PP'})\} \qquad (3.4)$$

$$\operatorname{cov}(T(P),T(P')) = \frac{G^2M^2}{rr'}\{\sum_{n=2}^{N}(\frac{a^2}{rr'})^n(2n+1)\ \varepsilon_n^2\ P_n(\cos\psi_{PP'})$$

$$+\sum_{n=N+1}^{\infty}(\frac{a^2}{rr'})^n\ (2n+1)\ \sigma_n^2\ P_n(\cos\psi_{PP'})\} \tag{3.5}$$

$$\operatorname{cov}(\Delta g(P),\Delta g(P')) = \frac{G^2M^2}{r^2r'^2}\{\sum_{n=2}^{N}(\frac{a^2}{rr'})^n(2n+1)(n-1)^2\ \varepsilon_n^2\ P_n(\cos\psi_{PP'})$$

$$+\sum_{n=N+1}^{\infty}(\frac{a^2}{rr'})^n\ (2n+1)(n-1)^2\ \sigma_n^2\ P_n(\cos\psi_{PP'})\} \tag{3.6}$$

where $$\varepsilon_n^2 = \sum_{m=0}^{n}(\varepsilon^2\overline{C}_{nm} + \varepsilon^2\overline{S}_{nm})/(2n+1) \tag{3.7}$$

and the unknown $\varepsilon^2\overline{C}_{nm}$, $\varepsilon^2\overline{S}_{nm}$ may be approximated by the estimated variances $e^2\overline{C}_{nm}$, $e^2\overline{S}_{nm}$ of the potential coefficients. σ_n is the r.m.s. variation of the potential coefficient by degree:

$$\sigma_n = [\sum_{m=0}^{n}(\overline{C}^2_{nm} + \overline{S}^2_{nm})/(2n+1)]^{\frac{1}{2}} \tag{3.8}$$

which is related to anomaly degree variance c_n by (Jekeli, 1978, p. 19):

$$c_n = \overline{\gamma}^2\ (n-1)^2\ (2n+1)\ \sigma_n^2 \tag{3.9}$$

The anomaly degree variances were computed by the 'two-component' global model, using the spherical approximation for the earth with radius $R = r = r' = 6371$ km:

$$c_n(R) = \alpha_1\frac{n-1}{n+A}\left(\frac{R_1^2}{R^2}\right)^{n+2} + \alpha_2\frac{n-1}{(n-2)(n+B)}\left(\frac{R_2^2}{R^2}\right)^{n+2} \tag{3.10}$$

with the parameters as determined by Rapp (1979, p. 15, Table 5, Case 1):

$$\alpha_1 = 3.4050 \text{ mgal}^2 \,, \qquad \alpha_2 = 140.03 \text{ mgal}^2$$

$$s_1 = \frac{R_1^2}{R^2} = 0.998006, \qquad s_2 = \frac{R_2^2}{R^2} = 0.914232 \tag{3.11}$$

$$A = 1 \qquad , \qquad B = 2$$

The advantage of modeling the degree variances by equations like (3.10) is that it allows the evaluation of infinite series in the second part of the equations (3.4) to (3.6) by using finite recursions (see Jekeli, 1978, Sec. III.4).

4. ACCURACY ESTIMATE OF PREDICTED DISTURBING POTENTIAL

We examine in this section the accuracy, $\sigma\varepsilon T$, of predicted T at a benchmark (SLR station) using the gravity anomalies in a cap of radius ψ. The variance-covariance matrix of errors in $\hat{T}(P_i) - \hat{T}(Q_j)$, i.e. $V(\hat{\varepsilon}\Delta T)$ in (2.5), will be discussed in Section 5. The evaluation of $\sigma\varepsilon T$ is described in Section 4.1, and the 'modified' anomalies Δg^* used for determining $\sigma\hat{\varepsilon}T$ are described in Section 4.2. The effect of anomaly density or separation, $\Delta\psi$, in a uniform pattern of anomalies, and the effect of the extent or spherical distance (radius) ψ of the anomalies on $\sigma\hat{\varepsilon}T$ is examined in Section 4.3. The effect of the errors in the low degree coefficients, and the effect of maximum degree of the normal gravitational field, on $\sigma\hat{\varepsilon}T$ is examined in Section 4.4. Finally the effect of the errors, $\sigma\varepsilon\Delta g^*$, in modified gravity anomalies, on $\sigma\hat{\varepsilon}T$ is examined in Section 4.5.

4.1 Disturbing Potential at Cap Centers

We may predict the disturbing potential $T(P_i)$ or $T(Q_j)$ at a cap center or station, P_i or Q_j, in principle, by using least squares collocation (Moritz, 1980, Sec. 14):

$$T(P_i) = \underline{C}_{TP_i,\Delta g}(\underline{\underline{C}}_{\Delta g,\Delta g} + \underline{\underline{D}})^{-1}\underline{\Delta g} = \underline{f}^T(\underline{z} + \underline{n}) \tag{4.1}$$

where

$$\underline{f} = (\underline{\underline{C}}_{zz} + \underline{\underline{D}})^{-1}\underline{C}^T_{Tz} \tag{4.2}$$

and the estimated accuracy, $\sigma\varepsilon T$, by:

$$\sigma^2\varepsilon T(P_i) = C_{TT} - \underline{f}^T\underline{C}_{TZ} \tag{4.3}$$

$C_{TP_i,\Delta g} \equiv \underline{C}_{Tz}$ is the covariance vector of the gravity anomalies vector Δg with $T(P_i)$, and its elements are given by (3.4). The gravity anomalies vector consists of a signal part, or true value $\underline{z}$, and uncorrelated random noise part $\underline{n}$. The variance covariance matrix of the signal is given

by $\underline{\underline{C}}_{zz} \equiv (\underline{\underline{C}}_{\Delta g, \Delta g})$; and that of the noise by $\underline{\underline{D}}$, which may be assumed to be a diagonal matrix, and computed as in (4.5) below. C_{TT} is the covariance $(T(P_i),T(P_i))$, which was assumed to be the same at all benchmarks (cap centers), when the anomalies have the same uniform pattern, and , for each cap. C_{TT} and elements of $\underline{\underline{C}}_{zz}$ may be computed by (3.5) and (3.6) respectively. The computation of elements of $\underline{\underline{C}}_{zz}$ and $\underline{C}_{Tz}$ are modified as in (4.6), (4.7) below when using ring averages of gravity anomalies.

The computations of covariances $\underline{C}_{Tz}$ and $\underline{\underline{C}}_{zz}$ are done much faster if the anomalies are located in a uniform pattern (Colombo, 1979), and the number of elements are much reduced if ring averages $\overline{\Delta g_k}$ are used instead of N_k individual anomalies Δg_{km} in the kth ring of specified radius around the cap center:

$$\overline{\Delta g_k} = \frac{1}{N_k} \sum_{m=1}^{N_k} \Delta g_{km} \tag{4.4}$$

$$d_{kk} = \frac{1}{N_k^2} \sum_{m=1}^{N_k} \sigma^2_{km} \tag{4.5}$$

and for simplicity, we assume for this study that $\sigma_{km} \equiv \sigma_{\Delta g}$ for all k and m. This simplifying assumption is not necessary for the computations of the actual vertical datum connections.

The covariances involving ring averages are related to covariances involving individual anomalies by:

$$\mathrm{cov}(\overline{\Delta g_k},\overline{\Delta g_l}) = \frac{1}{N_k N_l} \sum_{m=1}^{N_k} \sum_{n=1}^{N_l} \mathrm{cov}(\Delta g_{km},\Delta g_{ln}) \tag{4.6}$$

$$\mathrm{cov}(T(P_i),\overline{\Delta g_k}) = \frac{1}{N_k} \sum_{m=1}^{N_k} \mathrm{cov}(T(P_i), \Delta g_{km}) \tag{4.7}$$

There were an integer number of rings say N_p in a cap of radius , such that:

$$\Delta r = \;/N_p \qquad (4.7)^*$$

The distribution of anomalies in the rings was one anomaly at the cap center (zeroth ring); 6, 12 and 24 anomalies in the first, second, and third rings. The number of anomalies thereafter doubled as the radius of the ring doubled (i.e. in the 6th, 12th, 18th, etc. ring), and staying constant otherwise (see Colombo, 1980 a, p. 23). The prediction of $T(P_i)$ and $T(Q_j)$ was done from ring averages of gravity anomalies by utilizing (4.5) to (4.7).

It is assumed for this study that the 'observed' anomalies at separation of nearly, or slightly less than, $\Delta\psi$, and of extent slightly larger than $(\psi + \Delta\psi)$ are available, though these are not located in the uniform pattern described above; e.g. see NOAA (1982) for the conterminous United States. Additional gravity observations may be required for some caps. The data requirement is very modest, as will be discussed in Section 6.3. This set of available anomalies may then be used, as a first step, to predict anomalies at locations giving the uniform pattern of anomalies described above. We now discuss in Section 4.2 a set of 'modified' anomalies Δg^* in the uniform pattern; the modification, in principle, lies in using each cap center as the 'leveling datum' for all anomalies inside that cap. This modification bypasses the requirement of accurate physical realization of the geoid using coastal tide gages, i.e. the problem of sea surface topography.

4.2 Modified Gravity Anomalies

We define (Heiskanen and Moritz, Sec. 8.3) the gravity anomaly Δg at a point P' on the terrain from the observed gravity g:

$$\Delta g(P') = g(P') - \gamma(Q') \tag{4.8}$$

where γ is the normal gravity on the point Q' on the telluroid corresponding to P', such that

$$U(Q') + \Phi(Q') = W(P') \tag{4.9}$$

where $\Phi(Q') \approx \Phi(P')$ may be using the notation of (2.1). However, the

knowledge of $W(P')$ is subject to errors in the leveled potential differences $\Delta W_\ell(0,P') = W(0) - W(P')$ over long distances from the geoid potential $W(0)$ determined at the tide gages, with additional error due to the unknown sea surface topography.

To avoid this problem, Colombo (1980b, Sec. 3.3) proposed the use of 'modified' gravity anomaly $\Delta g^*(P')$ referenced to the normal gravity potential through the corresponding cap center P_i, such that:

$$\Delta g^*(P') = g(P') - \gamma(Q") \tag{4.8}^*$$

where $Q"$ is a point defined on a geopotential surface with the normal gravity potential

$$U(Q") + \Phi(Q") = U(P_i) + \Phi(P_i) - \Delta W_\ell(P_i,P') \tag{4.9}^*$$

which is well defined if we observe the leveled potential differences $\Delta W_\ell(P_i,P')$ over the short distances (P_i,P') inside a cap from the corresponding cap center. We thus need estimates of $\Delta W_\ell(P_i,P')$ for all gravity anomalies inside a cap to compute the modified gravity anomalies Δg^* in (4.8)*. Errors in ΔW_ℓ would propagate into errors into Δg^*. These would of course be much smaller than the corresponding errors in (4.8), due to shorter distances and will also be free of the errors of sea surface topography. The last statement is applicable as the normal gravity potential at the cap center, P_i, is computed from the SLR station coordinates.

We further note that as the right hand side of (4.9)* equals $W(P') - T(P_i)$, and assuming $\Phi(Q") \simeq \Phi(Q')$, we have using (4.9):

$$U(Q') = U(Q") + T(P_i) \tag{4.10}$$

Hence, $\Delta g^*(P') = \Delta g(P') + \gamma(Q') - \gamma(Q'') = \Delta g(P') - \frac{\partial\gamma}{\partial r}\frac{T(P_i)}{\gamma}$

$$= \Delta g(P') + \frac{2}{r(P_i)}\ T(P_i) \qquad (4.11)$$

as a spherical approximation. (4.11) permits a correction to be computed in (4.12) below to the estimated $T(P_i)$ from the modified anomalies, because in (4.11) Δg^* may be considered as the original anomalies with a bias term $kT(P_i)$, $k = 2/r(P_i)$. If Δg is in mgals, and T is in kgal.m, $k \approx (2 \times 10\ )/(6.371 \times 10^6) \approx 0.3$.

In (4.1) to (4.3), the covariances are based on Δg through (3.4) to (3.6) and (3.10). We now consider the effect on T and $\sigma\varepsilon T$, if the data consists of the modified anomalies Δg^* instead of Δg, but the covariances, and hence $\underline{f}$ in (4.2), is based on Δg. Let the estimate of $T(P_i)$ be denoted by $\tilde{T}(P_i)$ or $\hat{T}(P_i)$ based on the data $\overline{\Delta g}$ or $\overline{\Delta g^*}$ respectively, where the overbar indicates that N_q ring averages are used as in (4.4) and (4.7)*, $N_q = N_p + 1$. Then:

$$\tilde{T}(P_i) = T(P_i) + \tilde{\varepsilon}T(P_i) = \sum_{l=1}^{N_q} f_l \Delta g_l = \sum_{l=1}^{N_p} f_l \Delta g^*_l - kT(P_i)\sum_{l=1}^{N_q} f_l ,$$

i.e. $$T(P_i) + \frac{\tilde{\varepsilon}T(P_i)}{1+k\Sigma f_l} = \frac{\Sigma f_l \overline{\Delta g^*_l}}{1+l\Sigma f_l}$$

and as $\hat{T}(P_i) = T(P_i) + \hat{\varepsilon}T(P_i)$, we have finally:

$$\hat{\varepsilon}T(P_i) = \frac{\hat{\varepsilon}T(P_i)}{1+k\Sigma f_l}\ ,\ \hat{T}(P_i) = \frac{\Sigma f_l \overline{\Delta g^*_l}}{1+k\Sigma f_l}\ ,\ k = 2/r(P_i) \qquad (4.12)$$

where the summation extends over the N_q rings in the cap, and using (4.3):

$$\sigma^2\hat{\varepsilon}T(P_i) = (C_{TT} - \underline{f}^T\underline{C}_{TZ})/(1 + \frac{2}{r(P_i)}\sum_{l=1}^{N_q} f_l)^2 \qquad (4.13)$$

Henceforward, whenever we use $T(P_i)$ or $\sigma\hat{\varepsilon}T(P_i)$, it will be understood to mean $T(P_i)$ or $\sigma\hat{\varepsilon}T(P_i)$ based on modified gravity anomalies Δg^*, but all covariance would be based on Δg through the anomaly degree variance model (3.10) used for evaluating (3.4) to (3.6). The factor $(1 + k\Sigma f_1)$ and $(1 + k\Sigma f_1)^2$ is then required to be used as in (4.12) and (4.13) respectively.

The implementation of the above equations was done by slight modifications of the programs developed by Colombo (1980a, Appendix B).

4.3 Extent and Density of Gravity Anomalies

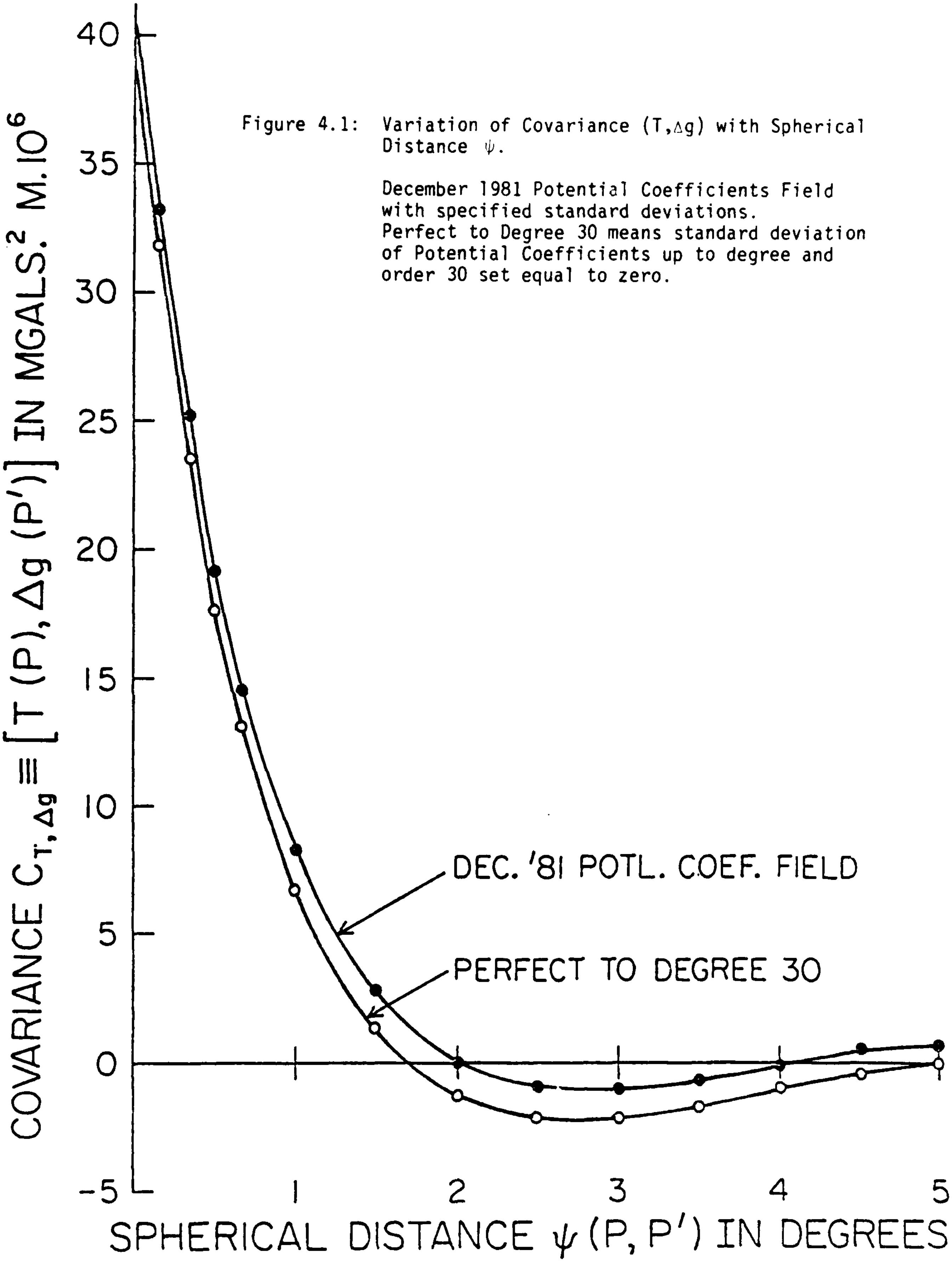
We now examine the effect of ψ, i.e. the cap radius or extent of gravity anomalies around the cap center; and the effect of $\Delta\psi$, i.e. the density or separation of anomalies, on the accuracy estimate $\sigma\hat{\varepsilon}T$. Colombo (1980 a,b) had considered $\psi = 5°$, but any reduction in ψ would make the realization of $\Delta g^*(P')$, with the needed estimates of $\Delta W_\ell(P_i,P')$ in (4.9)*, easier in practice.

Figure 4.1 shows a plot of the covariance $C_{T,\Delta g} = \text{cov}(T(P),\Delta g(P'))$ in (3.4) for variation in the spherical radius ψ up to 5°, where the normal gravitational field is represented by the Dec. '81 P.C. field ('imperfect' model) and also the 'perfect' model to N = 30 (see Section 3.2). The curves for coefficients perfect to N = 10,20 were not plotted to avoid congestion; they fall between the two plotted curves. The value of N in the expansion of T in (3.2) and the corresponding expansion of Δg in (3.3) was taken as 60. The use of this specific value of N will be discussed in Section 4.4. The plot of the covariance function $C_{T,\Delta g}$ would of course be more damped out if a still higher degree espansion is used in (3.2) and (3.3), i.e. when N is taken as 90, 120, 150, or 180. It is clear from Figure 4.1 that $C_{T,\Delta g}$ decreases rapidly with increase in ψ, and the extent of anomalies, for predicting T, may be reduced considerably from $\psi = 5°$ used by Colombo (1980 a,b).

For a specific value of ψ, we have also to consider the effect on σT for different density or separation, $\Delta\psi$, of the anomalies (See

Figure 4.1: Variation of Covariance (T,Δg) with Spherical Distance ψ.

December 1981 Potential Coefficients Field with specified standard deviations. Perfect to Degree 30 means standard deviation of Potential Coefficients up to degree and order 30 set equal to zero.



equations (4.13) and (4.7*)). The values of $\sigma\hat{\varepsilon}T$ are given in Table 4.1 for $\Delta\psi$ = 1/3°, 1/4°, 1/6° for some values of ψ. The total number of anomalies needed for some specific combinations of $\Delta\psi$ and ψ are also given. The normal gravitational field is described by Dec. '81 P.C. field to degree $N \leq 60$; and $\sigma\varepsilon\Delta g^*$ is 2 mgals. (The effect of variation in N and $\sigma\varepsilon\Delta g^*$ would be considered in Sections 4.4 and 4.5 respectively).

Table 4.1 Accuracy Estimate, $\sigma\hat{\varepsilon}T$ (Kgal.m), of Disturbing Potential at Cap Center for Various Density, $\Delta\psi$°, and Extent, ψ°, of Uniform Pattern of Modified Gravity Anomalies, Δg^*. $\sigma\varepsilon\Delta g^*$ = 2mgals; December '81 Potential Coefficients Field.

↓Δψ°	$\sigma\hat{\varepsilon}T$ (kgal.m)				↓Δψ°	Total # of Δg^* in a Cap			
ψ° →	2°	3°	4°	5°	ψ° →	2°	3°	4°	5°
1/3°			0.55	0.52	1/3°			475	763
1/4°		0.48	0.42	0.38	1/4°		475	859	1243
1/6°	0.51	0.41	0.34	0.28	1/6°	475	1051	1723	2875

Table 4.1 confirms the obvious expectation of lower $\sigma\hat{\varepsilon}T$ for a specific density $\Delta\psi$, as ψ is increased; and also of lower $\sigma\hat{\varepsilon}T$ for a specific ψ, as the density $\Delta\psi$ is increased. However, a large ψ along with a large $\Delta\psi$ is not suitable for realization in practice, as the total number of anomalies in a cap becomes very large, e.g. $\Delta\psi$ = 1/6°, ψ = 4° or 5°.

We can also see from Table 4.1 the possibility of an optimum combination of ψ and $\Delta\psi$. For example, ψ = 3°, $\Delta\psi$ = 1/6° gives about the same value of $\sigma\hat{\varepsilon}T$ as ψ = 4°, $\Delta\psi$ = 1/4°. Again, ψ = 2°, $\Delta\psi$ = 1/6° gives slightly higher $\sigma\hat{\varepsilon}T$ than ψ = 3°, $\Delta\psi$ = 1/4°; but in fact lower than $\Delta\psi$ = 1/3° for either ψ = 4° or 5°.

It is obviously preferable to have a smaller ψ, as we need the estimates of $\Delta W_{\ell}(P_i, P')$ for all anomalies in a data cap, and these can

be obtained with greater precision for smaller ψ. Additional tests were carried out to determine $\sigma\hat{\varepsilon}T$ for $\psi \leq 3°$, but with greater density of anomalies, i.e. $\Delta\psi \leq 1/6°$. The results are given in Table 4.2.

Table 4.2 Accuracy Estimate, $\sigma\hat{\varepsilon}T$ (kgal.m), of Disturbing Potential at Cap Center.
Extent $\psi \leq 3°$; $\Delta\psi \leq 1/6°$;
Other particulars as in Table 4.1.

$\downarrow\Delta\psi°$ / $\psi° \rightarrow$	$\sigma\hat{\varepsilon}T$ (kgal.m)				$\downarrow\Delta\psi°$ / $\psi° \rightarrow$	Total # of Δg^* in a Cap			
	0.5°	1°	2°	3°		0.5°	1°	2°	3°
1/6°	0.82	0.67	0.51	0.41	1/6°	43	139	475	1051
1/8°	0.82	0.66	0.50	0.40	1/8°	67	235	859	1723
1/12°	0.82	0.66			1/12°	139	475		

It is thus adequate to have the anomalies in a uniform pattern (4.7)* at a separation $\Delta\psi \simeq 10'$, as a more densely spaced set does not lead to significantly lower $\sigma\hat{\varepsilon}T$. We will examine the accuracy of vertical datum connection, specifically of Europe - U.S. connection, using ψ of 3°, 2° and 1° with a larger number of equations (2.3) being available as ψ is decreased.

The results in Table 4.1 and 4.2, and a few additional results, have been plotted in Figure 4.2.

4.4 Accuracy and Highest Degree of Normal Gravitational Field

It is of interest to know how much the accuracy estimate $\sigma\hat{\varepsilon}T$ would be improved if the low degree coefficients in the gravitational field were known to a higher accuracy than at present (Section 3.2). $\sigma\hat{\varepsilon}T$ is accordingly tabulated in Table 4.3 for the current December '81 field, as well as for the cases if the standard deviation of the coefficients is zero (perfect) to degree N = 10, 20 or 30. The extent of anomalies in a data cap, i.e. ψ, is varied; but as determined in Section 4.3, the anomaly separation $\Delta\psi$ in all cases is taken as $\Delta\psi = 1/6°$.

Figure 4.2: Variation of Accuracy Estimate $\sigma\hat{\epsilon}T$ of Disturbing Potential with Anomaly Cap Radius ψ and Anomaly Density $\Delta\psi$.

$\sigma\epsilon\Delta g^*$ = 2 mgals. December 1981 Potential Coefficients Field.
The Total number of anomalies in a data cap are given in Tables 4.1 and 4.2.

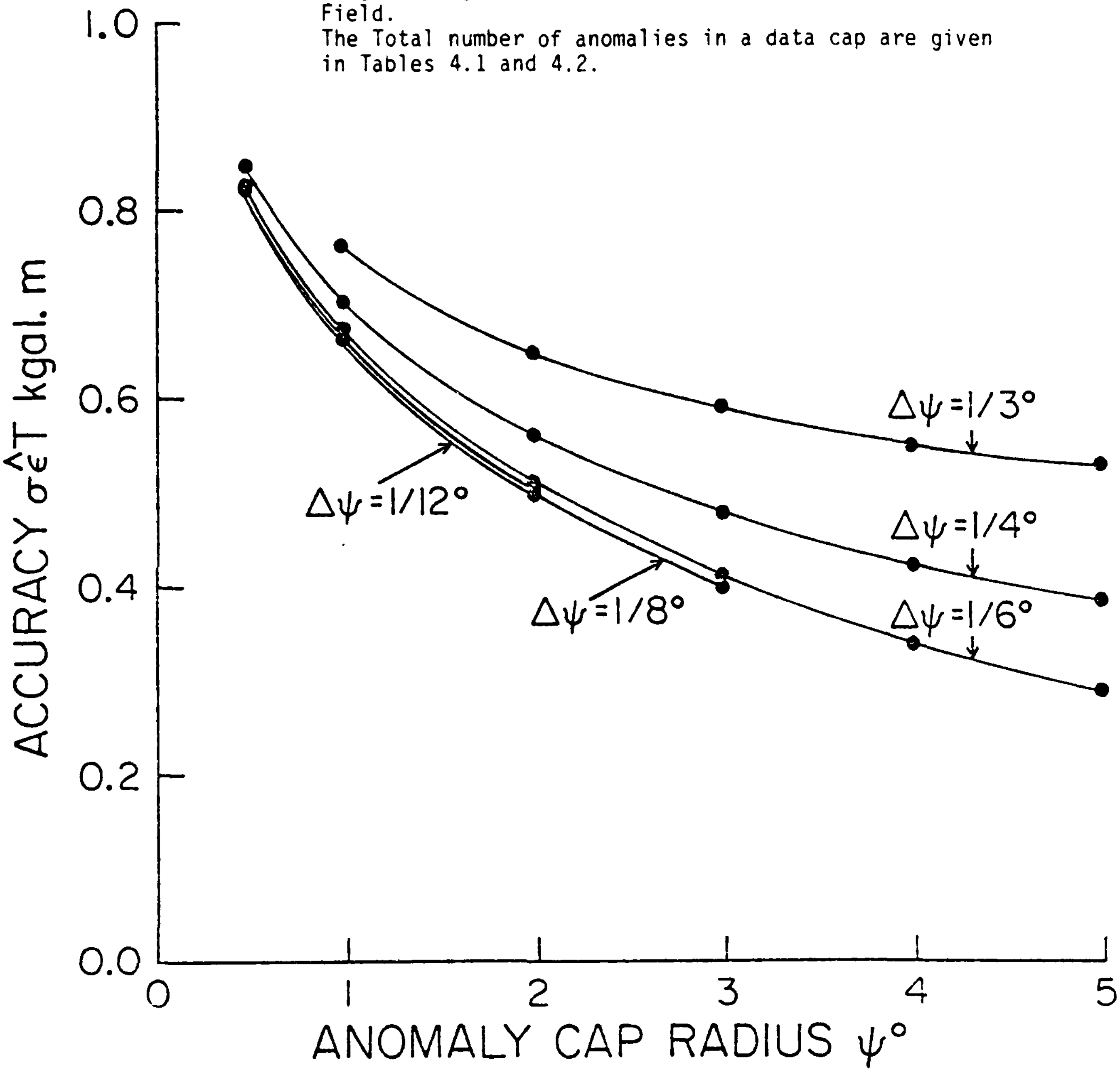


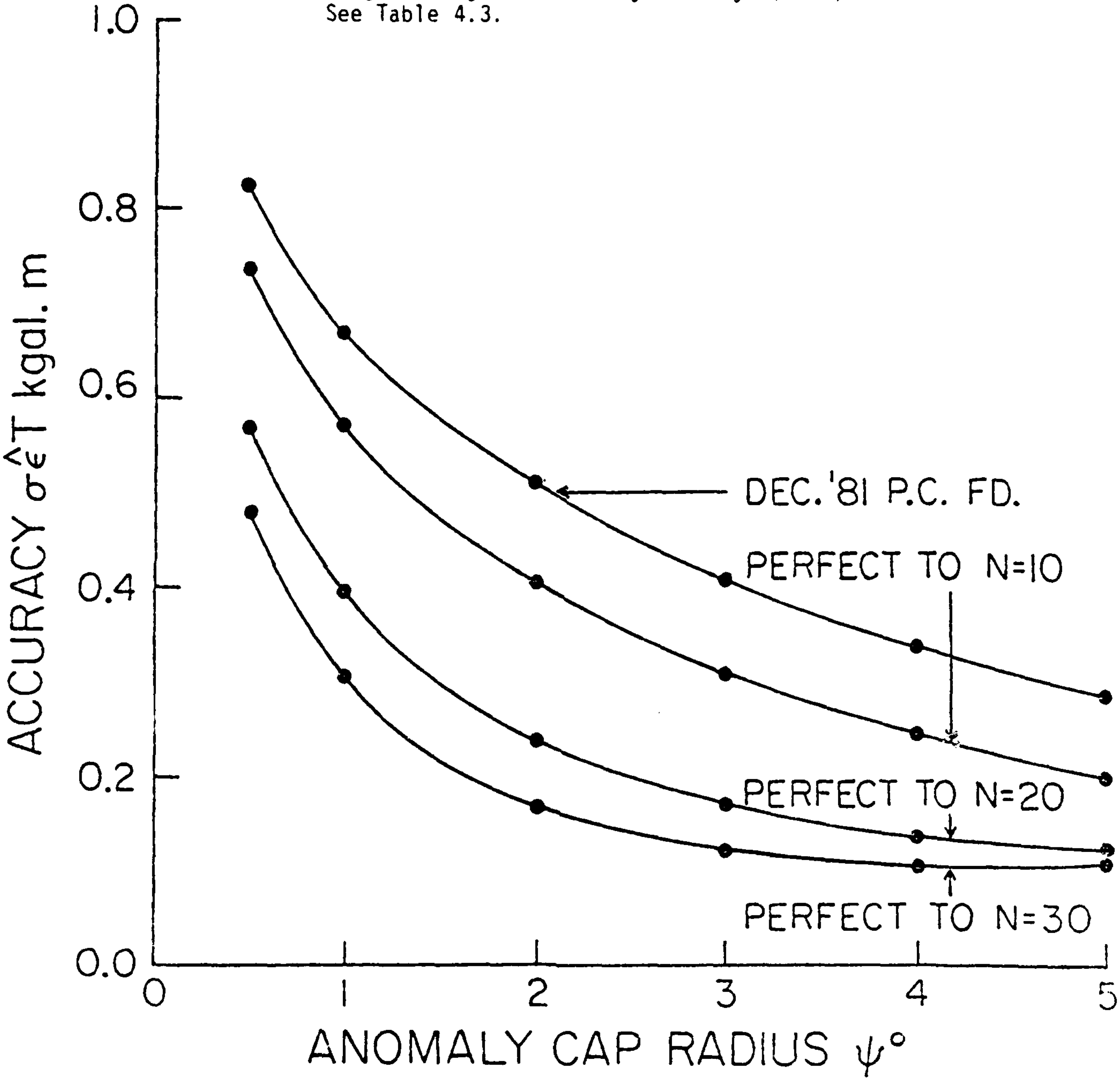
Table 4.3 Accuracy Estimate, $\sigma\hat{\varepsilon}T$ (kgal.m) of Disturbing Potential at Cap Center. Variation Due to Accuracy of Low Degree Normal Gravitational Field. Anomaly Separation, $\Delta\psi$ = 1/6°. $\sigma\varepsilon\Delta g^*$ = 2mgals

Accuracy of Low Degree Normal Field	$\sigma\hat{\varepsilon}T$ (kgal.m)			
	ψ=0.5°	ψ=1°	ψ=2°	ψ=3°
Dec. '81 Coefficients (e_n as in Table 3.2)	0.82	0.67	0.51	0.41
Perfect to N = 10 (e_n = 0 for N $\leq$ 10)	0.73	0.57	0.40	0.31
Perfect to N = 20 (e_n = 0 for N $\leq$ 20)	0.57	0.40	0.24	0.17
Perfect to N = 30 (e_n = 0 for N $\leq$ 30)	0.48	0.31	0.16	0.12

The results in Table 4.3, and some additional results for $\psi > 3°$, are plotted in Figure 4.3. We clearly see that for a given accuracy of the normal gravitational field, $\sigma\hat{\varepsilon}T$ improves as ψ is increased. However, $\sigma\hat{\varepsilon}T$ improves more strikingly for a given ψ, as the accuracy of low degree coefficients improves. We do not expect a perfect gravitational model in the near future, and all vertical datum connections in this report in Sections 6 and 7 are computed with the present accuracy of coefficients in the December '81 field. However, we may draw two conclusions from Figure 4.3. Firstly, the extent ψ of Δg^* may be reduced as the accuracy of low degree normal field improves; e.g., $\sigma\hat{\varepsilon}T$ of about 0.4 kgal.m may be achieved with anomalies in a data cap of 3°, 2° or 1° with Dec. '81 field, and a field perfect to N = 10, 20 respectively. Secondly, we may expect limiting values of $\sigma\hat{\varepsilon}T$ of about 0.3 kgal.m with data cap of ψ = 1°, which will improve to about 0.2 kgal.m with ψ = 2°. These improvements will come primarily from better knowledge of low degree coefficients; larger data caps ($\psi \geq 3°$) or greater data density ($\Delta\psi < 1/6°$) will not contribute significantly to lowering the accuracy ($\sigma\hat{\varepsilon}T$) of disturbing potential at cap centers.

Figure 4.3: Variation of Accuracy Estimate $\sigma\hat{\epsilon}T$ of Disturbing Potential with Accuracy of Low Degree Normal Gravitational Field.

$\sigma\epsilon\Delta g^*$ = 2 mgals. Anomaly Density $\Delta\psi$ = 1/6°.
See Table 4.3.



It is also of interest to examine the effect on σT in (4.13) of the maximum degree N of normal gravitational field used in defining T in (3.2); and the covariances (3.4) to (3.6) required in computing $\sigma_\varepsilon T$ in (4.3). This is shown in Figure 4.4 for the December '81 coefficients field, for anomaly separation at $\Delta\varphi = 1/6^\circ$, $\sigma_\varepsilon \Delta g^* = 2$ mgals; and for data cap radius $\psi = 0.5^\circ$, 1°, 2° and 3°. There is no significant difference in $\sigma_\varepsilon T$ in any of the cases considered, when the maximum degree N of normal gravitational field is increased beyond 60. This is due to the fact that the r.m.s. accuracy ε_n and the r.m.s. variation of the coefficient σ_n (defined in (3.7) and (3.8)), tend to be of the same order of magnitude when $N > 60$ (see Table 3.2). However, there is a marked difference in the computing time in the covariances (3.4) to (3.6) as the maximum degree N is increased. The second term in equations (3.4) to (3.6) is obtained by finite recursions with the use of the degree variance model (see end of Section 3.2); but the first term in equations (3.4) to (3.6) requires the summation of a series, which becomes expensive in computer time as N is increased. For $\psi = 3^\circ$, with 18 anomaly rings (4.4), the computing time for $\sigma_\varepsilon T$ for one data cap on the Amdahl V/6 computer was 13.4 seconds for $N = 60$, and 36.1 seconds for $N = 180$. However, as the number of anomaly rings are reduced to 12 for $\psi = 2^\circ$, the computing time for $\sigma_\varepsilon T$ was 8.7 seconds for $N = 60$, and 11.4 seconds for $N = 180$. The difference in computing time is further reduced when $\psi < 2^\circ$.

4.5 Accuracy of Gravity Anomalies

The accuracy estimate of modified gravity anomalies propagates into σT through (4.5) which defines the noise matrix used in computing f in (4.2) and $\sigma_\varepsilon T$ in (4.3); and finally through (4.13) we also take into account the underlying covariance model used for predicting the disturbing potential $T(P_i)$ or $T(Q_j)$ at cap centers in (4.12). This propagation to σT strictly refers to the random errors in gravity anomalies, which depends on the precision of gravity observations and the position, primarily the height, of the observation point. The computation of modified anomalies requires $\Delta W_i(P_i,P')$ observations through leveling in (4.9)* and (4.8)* from the cap center over distances of the order of $\psi = 3^\circ$, or smaller. Hence, a conservative estimate of 2 mgals was used for $\sigma_\varepsilon \Delta g^* = \sigma_{km}$ for

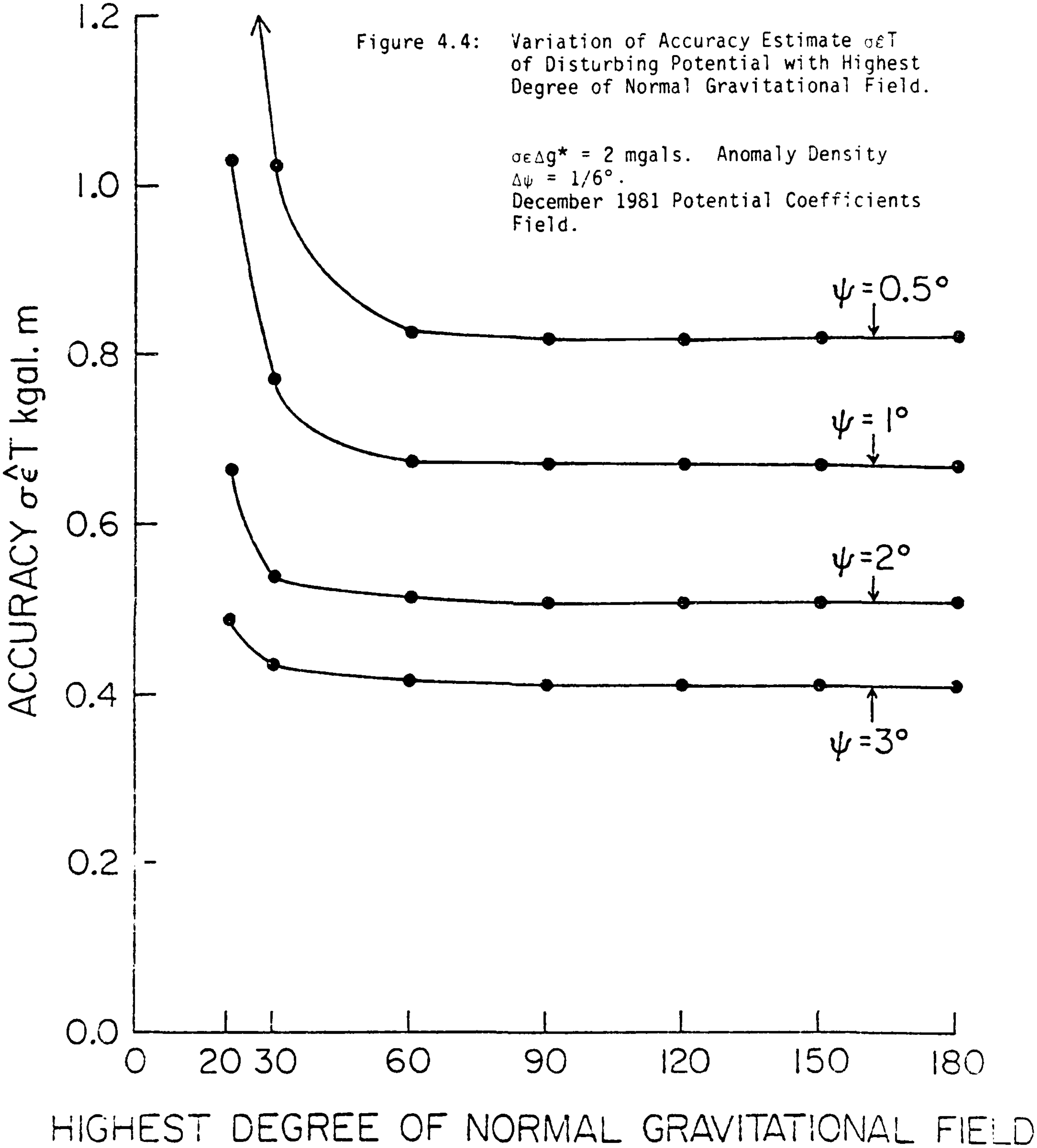


Figure 4.4: Variation of Accuracy Estimate $\sigma_{\epsilon T}$ of Disturbing Potential with Highest Degree of Normal Gravitational Field.

$\sigma_{\epsilon \Delta g^*}$ = 2 mgals. Anomaly Density $\Delta\psi = 1/6°$.
December 1981 Potential Coefficients Field.

all k and m in (4.5), i.e. for simplicity in this study, all gravity anomalies were assumed to have a standard deviation of 2 mgals. This estimate takes into account the gravity observation errors, the additional errors in leveling from the cap center and in the prediction of the uniform pattern of anomalies described below (4.7)*.

However, to examine the effect on $\sigma\hat{\varepsilon}T$ of larger $\sigma\varepsilon\Delta g^*$, several values of $\sigma\varepsilon\Delta g^*$ were tried up to 10 mgals for different cap sizes. These are tabulated in Table 4.4.

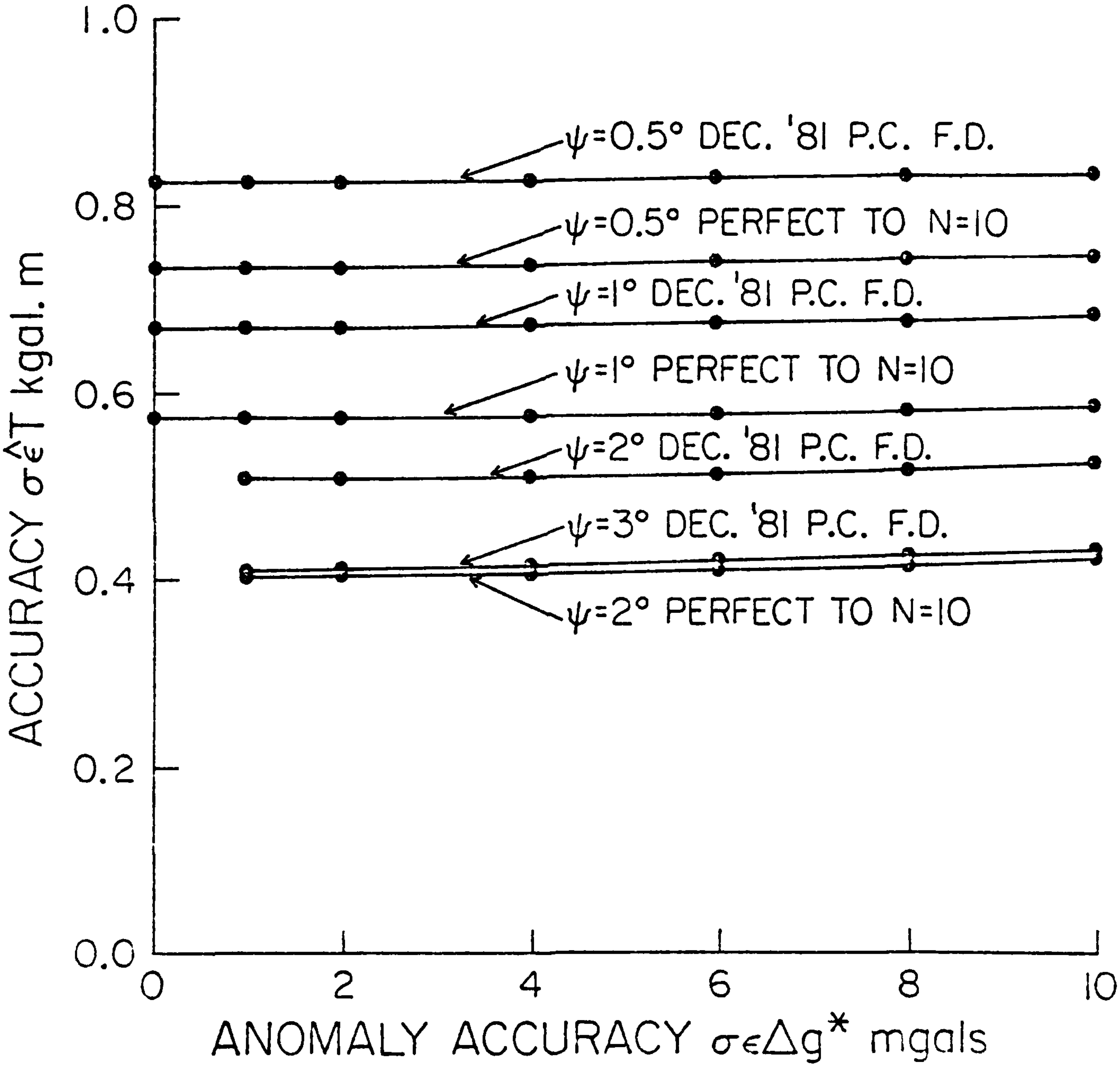
Table 4.4 Accuracy Estimate, $\sigma\hat{\varepsilon}T$ (kgal.m), of Disturbing Potential at Cap Center. Variation due to Accuracy Estimate, $\sigma\varepsilon\Delta g^*$, of Modified Gravity Anomalies. Uniform Pattern of Anomalies with Density $\Delta\psi$ = 1/6°. December '81 Potential Coefficient Field.

$\sigma\varepsilon\Delta g^*$	$\sigma\hat{\varepsilon}T$ (kgal.m)			
mgals	ψ = 0.5°	ψ = 1°	ψ = 2°	ψ = 3°
0	.823	.670		
1	.823	.670	.508	.410
2	.824	.670	.508	.411
4	.825	.672	.511	.414
6	.827	.674	.515	.422
8	.830	.678	.519	.429
10	.834	.683	.526	.437

The results in Table 4.4, and a few additional results for the case if the low degree ($N \leq 10$) potential coefficients were known perfectly, have been plotted in Figure 4.5. It is clear from Table 4.4 and Figure 4.5 that $\sigma\hat{\varepsilon}T$ depends strongly on the extent of gravity anomalies (with optimm density $\Delta\psi$ = 1/6°); and $\sigma\hat{\varepsilon}T$ is not particularly affected by the variation in $\sigma\varepsilon\Delta g^*$. This result is very helpful in answering the concern about the accuracy requirement of gravity anomalies over marine areas

Figure 4.5: Variation of Accuracy Estimate $\sigma\epsilon T$ of Disturbing Potential with Accuracy of Gravity Anomalies.

Anomaly Densiity $\Delta\psi$ = 1/6°.
See Table 4.4.



around those SLR stations, which are located in coastal areas (see Figure 3.1). Even if the random errors of some modified gravity anomalies are quite large, the accuracy estimate σT of disturbing potential at cap center is obtained to about ˙41, ˙51, ˙67 kgal.m for ψ = 3°, 2°, 1° respectively, if modified gravity anomalies are available in a uniform pattern at density $\Delta\psi$ = 1/6°.

A systematic error in gravity anomalies due to height error of as large as 50 cm is shown by Colombo (1980, p. 20, footnote) to cause an error of the order of 0˙05 kgal.m in εT, which may be neglected. A constant height error in all stations will result in a constant bias in all gravity anomalies. As the pattern of gravity anomalies, i.e. the extent $\psi°$ and density $\Delta\psi°$, is the same around each cap center, the resulting εT will be nearly the same and will largely cancel out in disturbing potential differences between cap pairs. For similar reasons, a systematic error in GM will largely cancel out from the normal and disturbing potential differences between cap pairs. The effect of random errors in positions of cap centers, and the geocentricity of the coordinate system, on $\sigma\Delta U$ was discussed in Section 2.2.

We now first examine in Section 5 the variance-covariance matrix of disturbing potential differences $V(\varepsilon\Delta T)$ in (2.5) between cap pairs, P_i, Q_j in regions A and B. The optimum extent ψ of gravity anomalies around each cap center is then examined in Section 6 by $\sigma\Delta W$(BMA,BMB) in (2.4) for the case of Europe - U.S.A. vertical datum connection, in view of larger number of equations being available as ψ is decreased below 3°.

5. VARIANCE COVARIANCE MATRIX OF DISTURBING POTENTIAL DIFFERENCES

The variance covariance matrix $V(\hat{\varepsilon}\Delta T)$ in (2.5) for errors in $T(P_i) - T(Q_j)$ are given by propagation of covariances through $\sigma^2\hat{\varepsilon}T(P_i)$ and $\sigma^2\hat{\varepsilon}T(Q_j)$ in (4.13) and by $\mathrm{cov}(\hat{\varepsilon}T(P_i), \hat{\varepsilon}T(Q_j))$. For example, the diagonal element v_{kk} of $V(\hat{\varepsilon}\Delta T)$ is given by:

$$v_{kk} = M\{(\hat{\varepsilon}T(P_{ik}) - \hat{\varepsilon}T(P_{jk}))^2\}$$

$$= \sigma^2\hat{\varepsilon}T(P_{ik}) + \sigma^2\hat{\varepsilon}T(P_{jk}) - 2M\{\hat{\varepsilon}T(P_{ik})\ \hat{\varepsilon}T(Q_{jk})\}, \tag{5.1}$$

where M is the averaging operator, and $k = 1,\ldots, N_e$ (see equation 2.3)).

Similarly, the off diagonal element v_{kl} of $V(\hat{\varepsilon}\Delta T)$ is given by

$$v_{kl} = M\{(\hat{\varepsilon}T(P_{ik}) - \hat{\varepsilon}T(Q_{jk}))\ (\hat{\varepsilon}T(P_{il}) - \hat{\varepsilon}T(Q_{jl}))\}$$

$$= M\{\hat{\varepsilon}T(P_{ik})\ \hat{\varepsilon}T(P_{il})\} + M\{\hat{\varepsilon}T(Q_{jk})\ \hat{\varepsilon}T(Q_{jl})\} - M\{\hat{\varepsilon}T(P_{ik})\ \hat{\varepsilon}T(Q_{jl})\}$$

$$- M\{\hat{\varepsilon}T(P_{il})\ \hat{\varepsilon}T(Q_{jk})\} \tag{5.2}$$

and a representative term $M\{\hat{\varepsilon}T(P_{ik})\ \hat{\varepsilon}T(Q_{jl})\} \equiv M\{\hat{\varepsilon}T(P_i)\ \hat{\varepsilon}T(Q_j)\}$ for convenience of notation, and using (4.12), is given by:

$$M\{\hat{\varepsilon}T(P_i)\ \hat{\varepsilon}T(Q_j)\} = M\left\{\frac{T(P_i) - f_i^T d_i}{1+ \frac{2}{r(P_i)}\ f_i} \cdot \frac{T(Q_j) - f_j^T d_j}{1 + \frac{2}{r(Q_j)}\ f_j}\right\}$$

$$= \left[C_{TT}(P_i,Q_j) - 2C^T_{Td}(P_i,Q_j)f + f^T C_{d_i d_j} f\right](1+ \frac{2}{r(P_i)}\ f_i)^{-1}(1+ \frac{2}{r(Q_j)}\ f_j)^{-1} \tag{5.3}$$

where d are the vectors of ring averages of modified gravity anomalies; other notations are similar to (4.1) to (4.3); and we have assumed in (5.3) that the pattern of anomalies, and their noise matrix, is the same around each cap center.

The computation of covariances $\underline{\underline{C}}d_i d_j$ of ring averages of gravity anomalies in (5.3) is very expensive on computer time, if these are computed rigorously by (4.6) and (3.6). It is adequate for (5.3) to compute the covariance of ring averages k and l, i.e. $M\{\overline{\Delta g}_{ik}\overline{\Delta g}_{jl}\}$ by Colombo (1980, p. 31, (4.17)):

$$M\{\overline{\Delta g}_{ik}\ \overline{\Delta g}_{jl}\} = M\{\frac{1}{2\pi a\ \sin\psi_k}\int_0^{2\pi}\Delta g(\psi_k, \alpha)d\alpha \cdot \frac{1}{2\pi a\ \sin\psi_l}\int_0^{2\pi}\Delta g(\psi_l, \alpha)d\theta\}$$

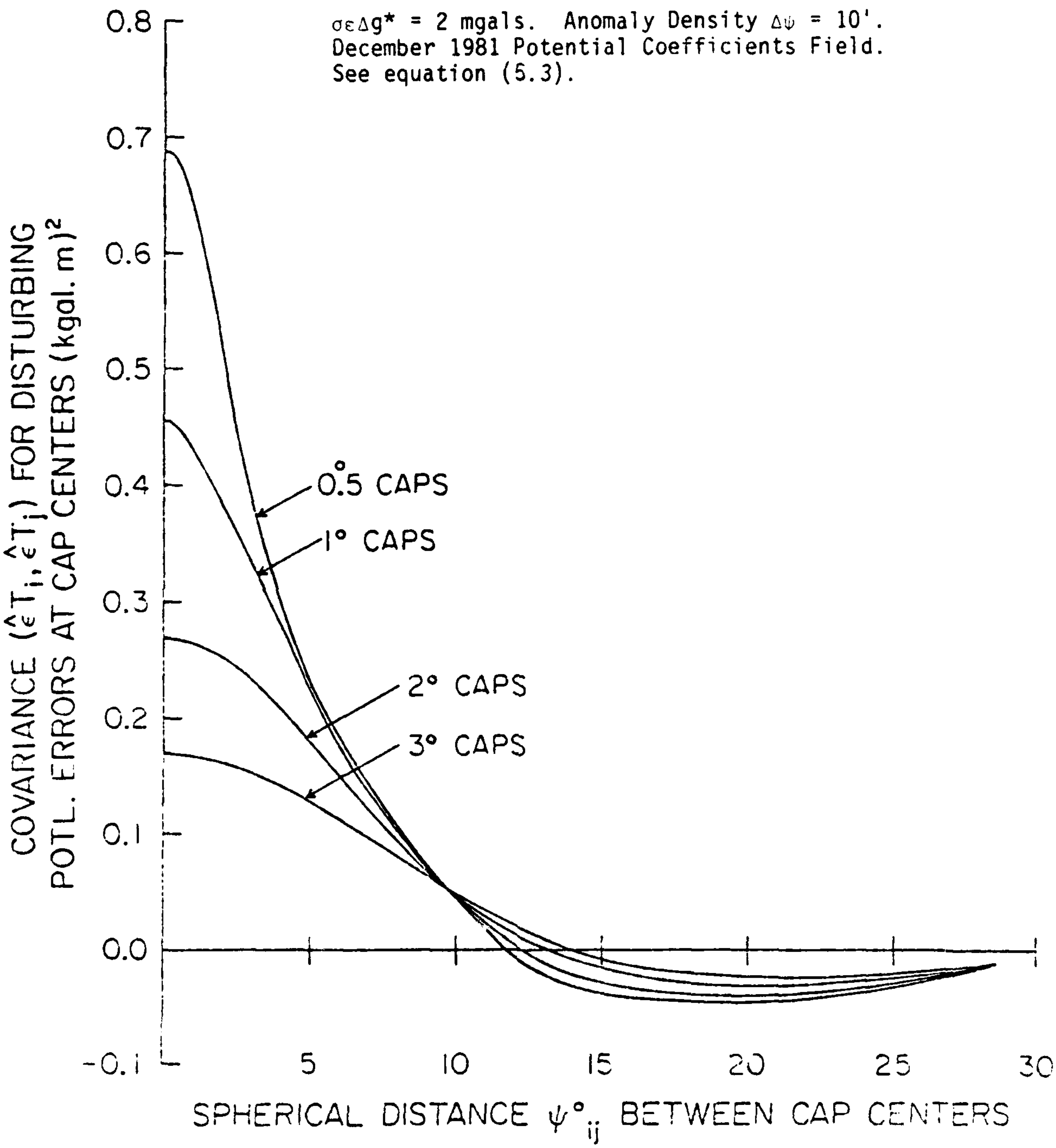
$$= \frac{G^2M^2}{a^4}\{\sum_{n=2}^{N}(2n+1)(n-1)^2\ \sigma_n^2\ P_n(\cos\psi_k)\ P_n(\cos\psi_l)\ P_n(\cos\psi_{ij})$$

$$+\sum_{n=N+1}^{N_{max}}(2n+1)(n-1)^2\ \sigma_n^2\ P_n(\cos\psi_k)\ P_n(\cos\psi_l)\ P_n(\cos\psi_{ij})\} \qquad (5.4)$$

where ψ_k, ψ_l are the radii for the rings k and l, ψ_{ij} is the spherical distance between cap centers i and j, and other notations are as in (3.6).

The covariances for disturbing potential at cap centers from (5.3) have been plotted in Figure 5.1 against spherical distance ψ_{ij} when all caps are of a given size, i.e. 3°, 2°, 1°, 0.5°. December 1981 Potential coefficients field was used with $N = 60$ and $N_{max} = 500$. The elements of the matrix $V(\epsilon\Delta T)$ for disturbing potential differences are computed according to (5.1) and (5.2) based on the cap pairs P_i, Q_j for the vertical datum connection equations (2.1).

Figure 5.1: Covariances Between Cap Center Disturbing Potential Errors.



6. ACCURACY ESTIMATE OF EUROPE-USA VERTICAL DATUM CONNECTION

We note from Figure 3.1 that the maximum number of satellite laser ranging (SLR) stations, whose coordinates are available in the SL4 coordinate system (Table 3.1) and which are, or may be, interconnected by leveling in a region, are 14 in USA and 4 in Europe. We therefore examine in this section the various possibilities of achieving Europe-USA vertical datum connection by estimating the disturbing potential from modified anomalies in various capsizes around the SLR stations. The effect of errors in the geocentricity of the SL4 coordinate system, the effect of random errors in SLR station positions, and the effect of leveling errors inside each region on the accuracy estimate of the vertical datum connection is examined in Section 6.2. We then discuss in Section 6.3 the optimum capsize for Europe-USA vertical datum connection.

6.1 SLR Stations for Vertical Datum Connection

Figures 6.1 and 6.2 show the SLR stations available in Europe and USA for the vertical datum connection. A cap of radius $\psi = 1°$ has been drawn around each cap center, in which ' modified' gravity anomalies Δg^* (Section 4.2) may be used, at a spacing of 10' ($\Delta\psi = 1/6°$) in a uniform pattern (see below (4.7)*), to predict $\hat{T}(P_i)$ or $\hat{T}(Q_j)$. With $\psi = 1°$, we have 4 'benchmarks' P_i in Europe and 14 'benchmarks' Q_j in USA to establish the vertical datum connection, by using $N_e = 17$ independent equations (2.1). To ensure that these equations are linearly independent (see (2.3)), and to ensure numerical stability in the solution of $\Delta W(BMA,BMB)$, the caps should not be overlapping so that different sets of gravity anomalies are used to predict T at each cap center.

We recall from line 1 of Table 4.2 that the accuracy estimate $\sigma\hat{T}$ of the disturbing potential at cap center improves as the cap radius ψ is increased. However, the number of caps then becomes smaller to avoid the overlapping of caps. Figures 6.3 and 6.4 show the available cap centers in USA when the radius ψ is 2° and 3° respectively. The available cap centers in USA are therefore reduced from 14 to 10 and 5, as ψ is increased from 1° to 2° and 3°. All the 4 caps in Europe remain

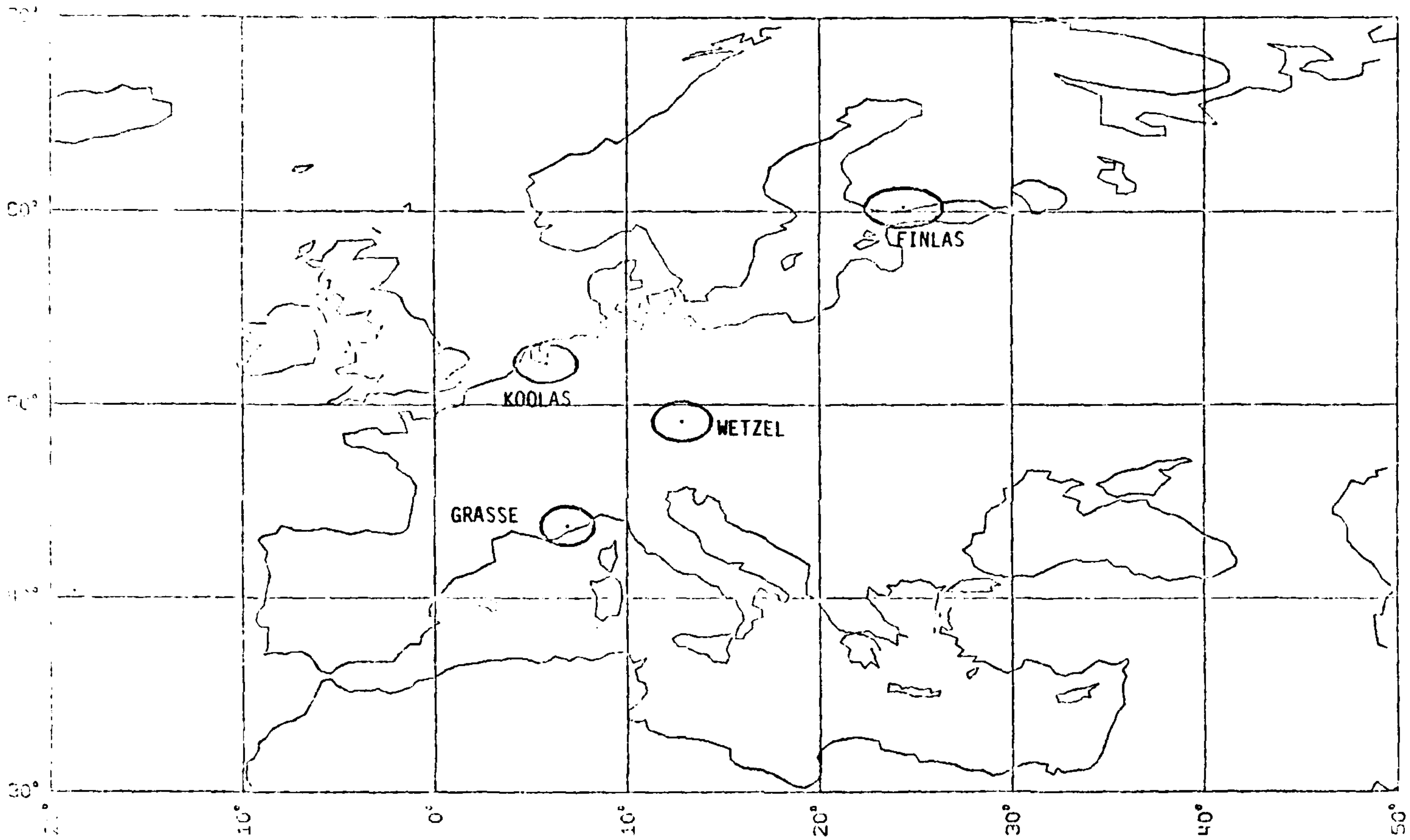


Figure 6.1: Satellite Laser Ranging Stations in Europe in the SL4 System (Smith et al., 1982). Gravity Anomaly Δg^* Cap Radius $\psi = 1°$ for Predicting Disturbing Potential at Cap Centers. Number of Caps = 4 (see Table 3.1 for details).

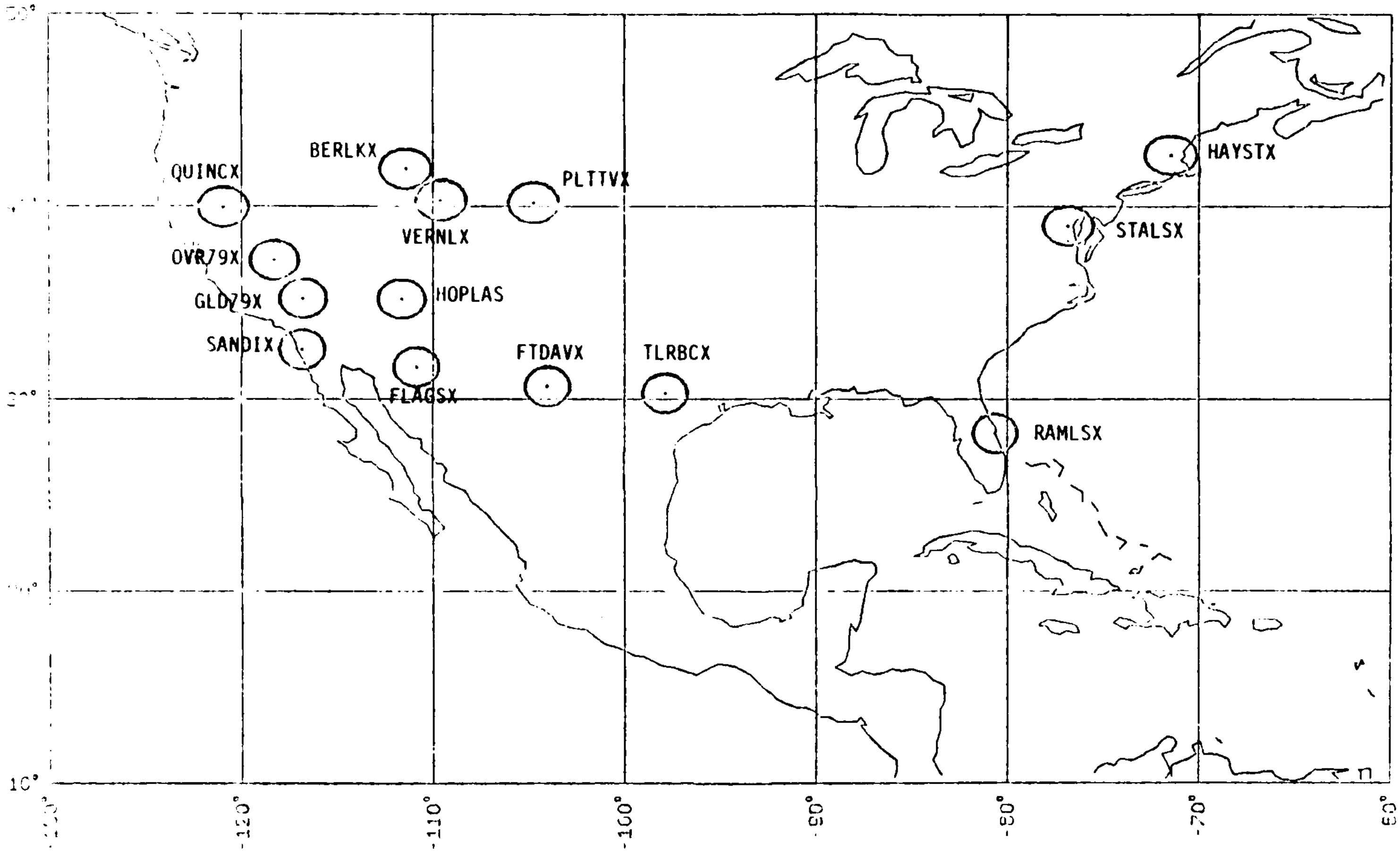


Figure 6.2: Satellite Laser Ranging Stations in USA in the SL4 System (Smith et al., 1982). Gravity Anomaly Δg^* Cap Radius $\psi = 1°$ for Predicting Disturbing Potential at Cap Centers. Number of Caps = 14 (see Table 3.1 for details).

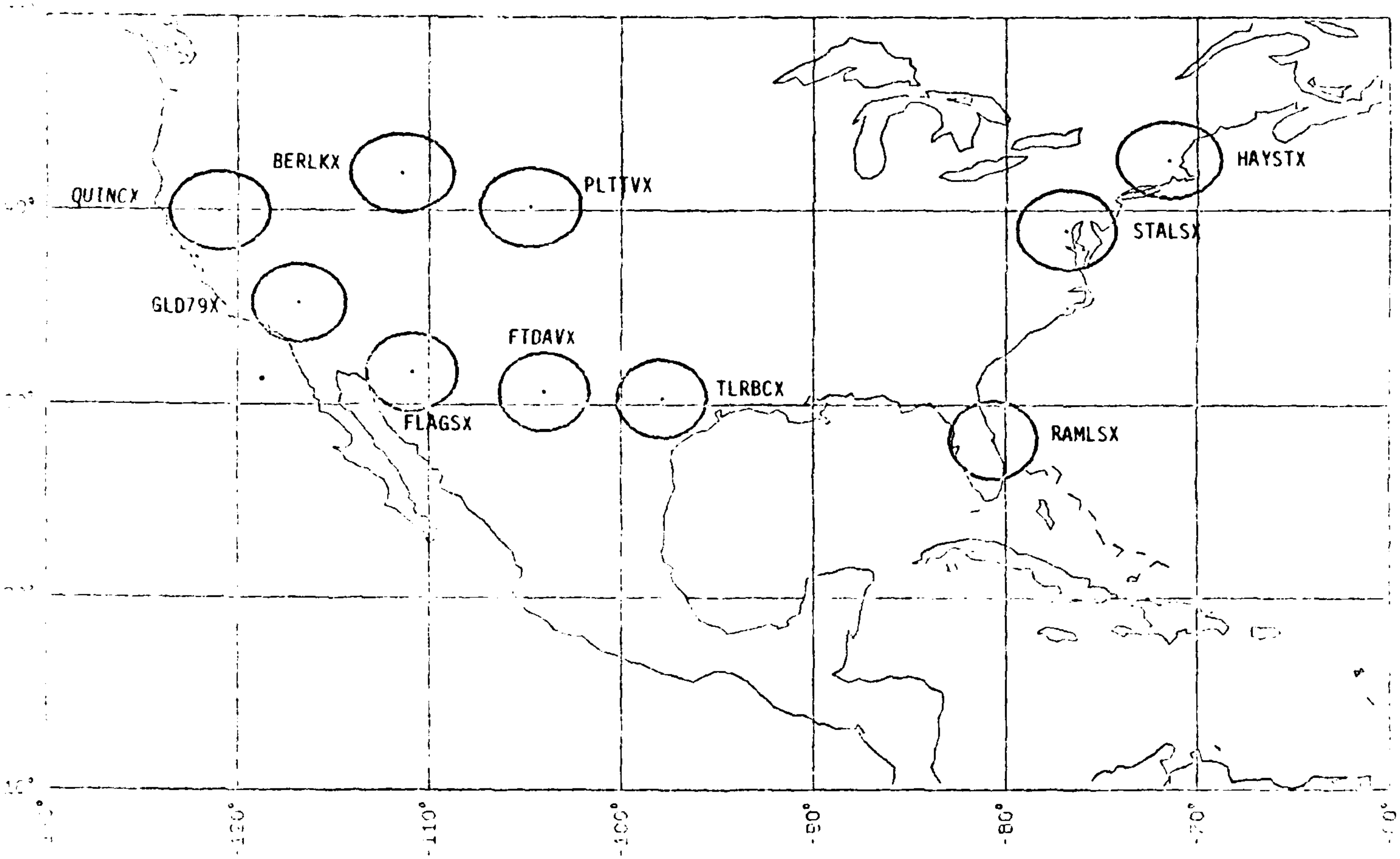


Figure 6.3: Cap Centers in USA for Vertical Datum Connection. Cap Radius ψ = 2°. Number of caps = 10 (Table 6.1).

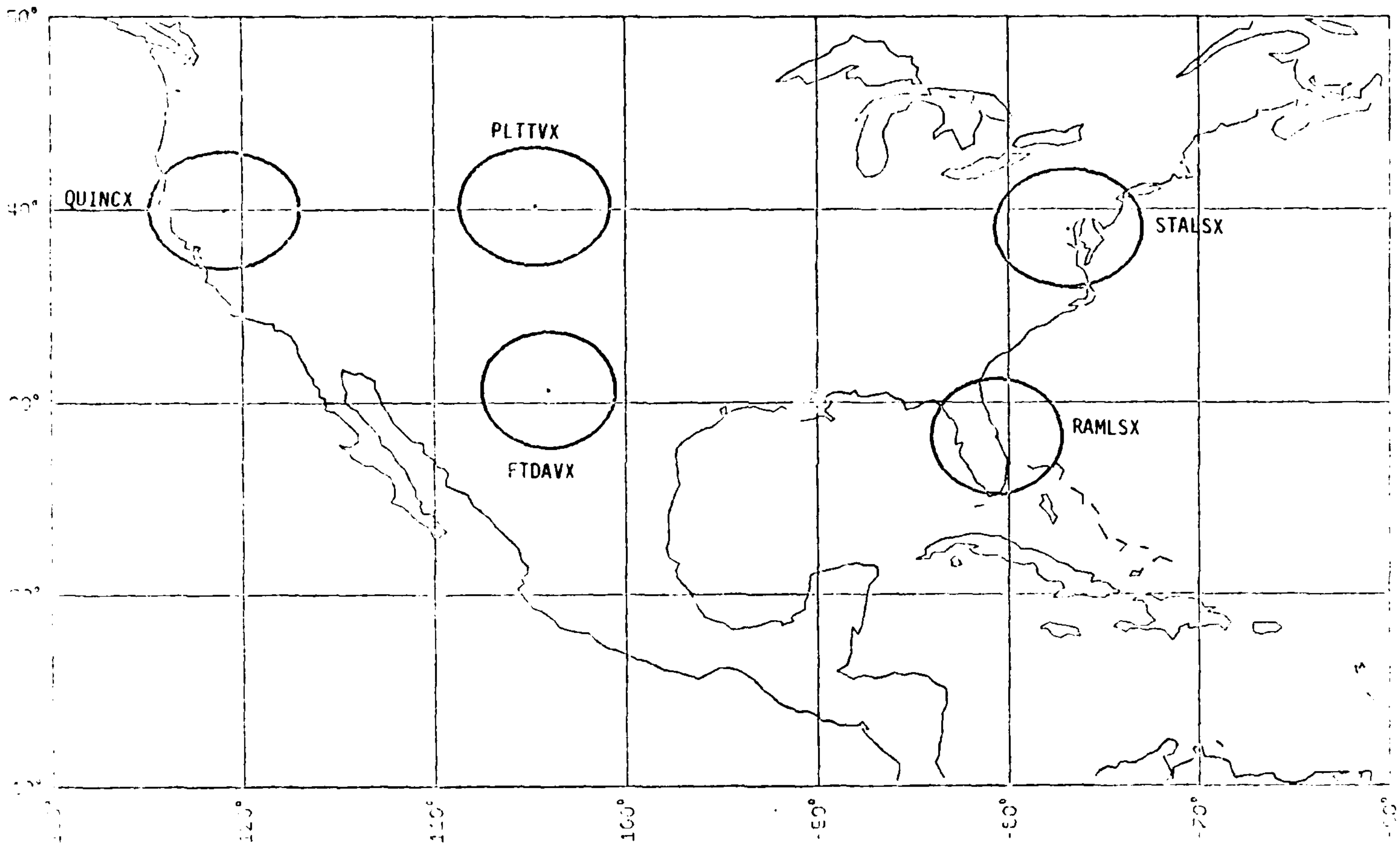


Figure 6.4: Cap Centers in USA for Vertical Datum Connection. Cap Radius ψ = 3°. Number of Caps = 5 (Table 6.1).

non-overlapping even with $\psi = 3°$, except for a very slight overlap between the caps around WETZEL and KOOLAS. The number of equations (2.1) for the vertical datum connection are thus N_e = 17, 13, 8 in the three cases. The advantage of increased accuracy in the predicted $\hat{T}(P_i)$ or $T(Q_j)$ is therefore somewhat offset by the reduction in the number of equations N_e in (2.3). Also, the increase in ψ very sharply increases the number of modified anomalies (Table 4.2, line 1, last 4 columns) required in each cap; and accordingly increases the field work for determining the potential differences $\Delta W_2(P_i,P')$ in (4.9)* from the cap centers to the anomalies.

The number of linearly independent equations (2.1) are one less than the total number of caps in both regions. The choice of cap pairs P_i, Q_j in (2.1) is arbitrary so long as these equations are independent; the actual choice is best made for the simplicity of vertical datum connection. For example, when $\psi = 3°$, with 4 caps P_i in Europe and 5 caps Q_j in USA, the cap pairs for forming equations (2.1) for the vertical datum connection may be as follows (see Table 3.1 for details of stations):

P_i: WETZEL GRASSE KOOLAS FINLAS

Q_j: PLTTVX QUINCX FTDAVX STALSX RAMLSX

There are 8 linearly independent equations, which are indicated by a line joining the two benchmarks, one in each region, i.e. P_i and Q_j between which (2.1) is formed. Here, WETZEL and PLTTVX have been selected as BMA and BMB, again rather arbitrarily but for the consideration of being centrally placed in each region (Figures 6.1 and 6.4) to give optimum lengths for leveling from BMA or BMB to cap centers. The central location of BMA and BMB is convenient, as we have to consider potential differences $\Delta W_1(\mathrm{BMA},P_i)$ and $\Delta W_1(\mathrm{BMB},Q_j)$ by leveling to enable each of the 8 equations (2.1) being written in terms of $\Delta W(\mathrm{BMA},\mathrm{BMB})$.

From Figure 6.1, it is reasonable to assume that $\Delta W_{i}(BMA,P_{i})$ may be obtained by independent routes from BMA to each cap center P_i, and thus $V(\sigma\Delta W_i)_A$ would be a diagonal matrix. From Figure 6.4, if we assume that the route for leveling from PLTTVX to RAMLSX is through STALSX, then $V(\sigma\Delta W_i)_B$ would be as in (2.13).

The particulars for the Europe - USA vertical datum connection for $\psi = 3°, 2°, 1°$ are given in Table 6.1. The accuracy estimate $\sigma\Delta W(BMA,BMB)$ is also given for the three cases. To highlight the effect of differing number of equations for the three values of ψ, $\sigma\Delta W$ is quoted without considering any geocentricity errors in SL4 coordinate system, and without considering any station position errors or any leveling errors. The effect of these later errors is examined in Section 6.2. The normal gravitational field is described by December 1981 potential coefficients (Rapp, 1981). The standard deviations of the potential coefficients is considered in determining $\sigma\hat{\delta}T$ (see equation (3.2)). The standard deviation, $\sigma\varepsilon\Delta g^*$, of modified gravity anomalies, is considered as 2 mgals. The modified gravity anomalies were considered in a uniform pattern (see below (4.7)*) at a density $\Delta\psi = 10'$.

We note from Table 6.1 that because of greater number of caps being available for establishing the vertical datum connection, anomaly data in capsize of $\psi = 1°$ gives $\sigma\Delta W(BMA,BMB)$ of 0.45 kgal.m, which is slightly worse than 0.37 kgal.m for $\psi = 2°$. But, considering that $\Delta W_i(P_i,P')$ observations by leveling are required from cap center to much lesser number of anomalies in each cap (of the order of 150 anomalies for $\psi = 1°$ to the order of 500 for $\psi = 2°$), capsize of $\psi = 1°$ appears to be adequate for establishing the vertical datum connection. In fact, if we use $\psi = 0.5°$ and even if we do not have additional caps, we get $\sigma\Delta W(BMA,BMB) = 0.54$ kgal.m, which is much better than the present likely accuracy of 1.41 kgal.m for connection between coastal benchmarks in the two regions if we assume the r.m.s. value of sea surface topography to be about 1 meter.

We now examine the effect on the Europe - USA vertical datum connection of other errors, which were not considered in Table 6.1.

Table 6.1 Europe-USA Vertical Datum Connection for Various Capsizes ψ^0. Normal Gravitational Field, December 1981 Potential Coefficients. Modified Anomalies Accuracy $\sigma_{\Delta g^*}$ = 2mgals. Anomaly Density 10'. No other errors considered.

Anomaly Cap Radius ψ^0	3°	2°	1°
P_i: Cap Centers in Europe (See No. System Below)	1, 2, 3, 4	1, 2, 3, 4	1, 2, 3, 4
Q_j: Cap Centers in USA (See No. System Below)	1, 4, 8, 12, 14	1, 3, 4, 6, 8, 9, 11 12, 13, 14,	1-14
BM A	WETZEL	WETZEL	WETZEL
BM B	PLTTVX	PLTTVX	PLTTVX
# Equations for Datum Connection	8	13	17
Cap Pairs for Datum Connection	1 1 2 2 3 3 4 4 1 4 4 8 8 12 12 14	1 1 1 2 2 2 3 3 3 3 1 3 4 4 6 8 8 9 11 12 4 4 4 12 13 14	1 1 1 1 2 2 2 2 2 3 1 2 3 4 4 5 6 7 8 8 3 3 3 3 4 4 4 9 10 11 12 12 13 14
Leveling Connections (In USA)	1-2, 1-3, 1-4, 1-4-5	1-3, 1-3-4, 1-3-4-6, 1-8, 1-8-9, 1-8-11, 1-12, 1-12-13, 1-12-14	1-2, 1-2-3, 1-2-4, 1-2-5 1-2-5-6, 1-2-5-6-7, 1-8, 1-8-9, 1-8-9-10, 1-8-11, 1-12, 1-12-13, 1-12-14
# Anomaly Ring Averages/Cap	19	13	7
# Anomalies per Cap	1051	475	139
Accuracy of Predicted T in a cap (σ_T - kgal.m)	0.41	0.51	0.67
Accuracy Estimates of Datum connection ($\sigma_{\Delta W(BMA,BMB)}$-kgal.m)	0.31	0.37	0.45

No. System for P_i: (1) WETZEL; (2) GRASSE; (3) KOOLAS; (4) FINLAS
No. System for Q_j: (1) PLTTVX; (2) VERNLX; (3) BERLKX; (4) QUINCX; (5) OVR79X; (6) GLD79X; (7) SANDIX (8) FTDAVX; (9) HOPLAS; (10) FLAGSX; (11) TLRBCX; (12) STALSX; (13) HAYSTX; (14) RAMLSX

See Table 3.1 for detailed information on these SLR stations.

6.2 Variations Due to Coordinate System, Position and Leveling Errors

If the origin of the SL4 coordinate system does not coincide with the center of mass (geocenter) of the earth, it results in errors in the positions of cap centers, whose radial component is given by (2.9). This causes additional covariances given by (2.10) in the variance covariance matrix of normal gravitational potential differences $V(\varepsilon\Delta U)$ of (2.5). The effect on $\sigma\Delta W(BMA,BMB)$ was examined for different uncertainties in the shifts of the origin of SL4 coordinate system from the geocenter, as characterized by the standard deviation, σs, of the shift. σs was varied from 0 to 50 cm.

Besides the shift of the origin, the determination of station positions have errors with respect to the SL4 coordinate system. The error in the normal gravitational potential is primarily caused by the radial component of the error in station position, and is given by (2.7). Assuming no correlation between radial errors of different cap centers, the variances in $V(\varepsilon\Delta U)$ are given by (2.8). The effect on $\sigma\Delta W(BMA,BMB)$ was examined for different estimates of radial errors, as characterized by their standard deviation, $\sigma\varepsilon r$. $\sigma\varepsilon r$ was also varied from 0 to 50 cm.

We also consider the effect on $\sigma\Delta W(BMA,BMB)$ of leveling errors in estimating potential differences $\Delta W_\ell(BMA,P_i)$ and $\Delta W_\ell(BMB,Q_j)$ in relating potential differences between cap pairs P_i, Q_j to $\Delta W(BMA, BMB)$ through (2.1). The formation of $V(\varepsilon\Delta\Delta W_\ell)$ in (2.5) was discussed in Section 2.3 by summation of corresponding elements in $V(\varepsilon\Delta W_\ell)_A$ and $V(\varepsilon\Delta W_\ell)_B$ in each region. The latter matrices are obtained in the form of (2.13) depending on the routes of leveling connections from BMA (and BMB) to cap centers P_i (and Q_j). The leveling connections for the Europe - USA vertical datum connection were discussed in Section 6.1, and specifically shown in Table 6.1. The estimate of errors in ΔW_ℓ due to random errors in leveling is given by (2.12):

$$\sigma\Delta W_\ell = k(0.1\sqrt{\ell})\ \text{kgal.m},\ \ell \text{ in } 10^3 \text{ km}$$

The effect on $\sigma\Delta W$(BMA,BMB) was examined for different accuracy estimates for leveling by varying k from 0 to 4.

The accuracy estimate $\sigma\Delta W$(BMA,BMB) for the Europe - USA vertical datum connection is tabulated in Table 6.2 for the variations discussed in this section. The variation in $\sigma\Delta W$ is tabulated for each of the three errors separately for the cases of $\psi = 0.5°$, $1°$, and $2°$.

We note from Table 6.2 that the main variation in $\sigma\Delta W$(BMA,BMB) is due to the anomaly data cap size as discussed in Section 6.1. The effect of uncertainties in the shift of origin (σs) of the SL4 system, and the effect of uncertainties in radial position ($\sigma\varepsilon r$), on $\sigma\Delta W$ is comparatively smaller. The effect of σs is slightly larger as compared to [illegible].

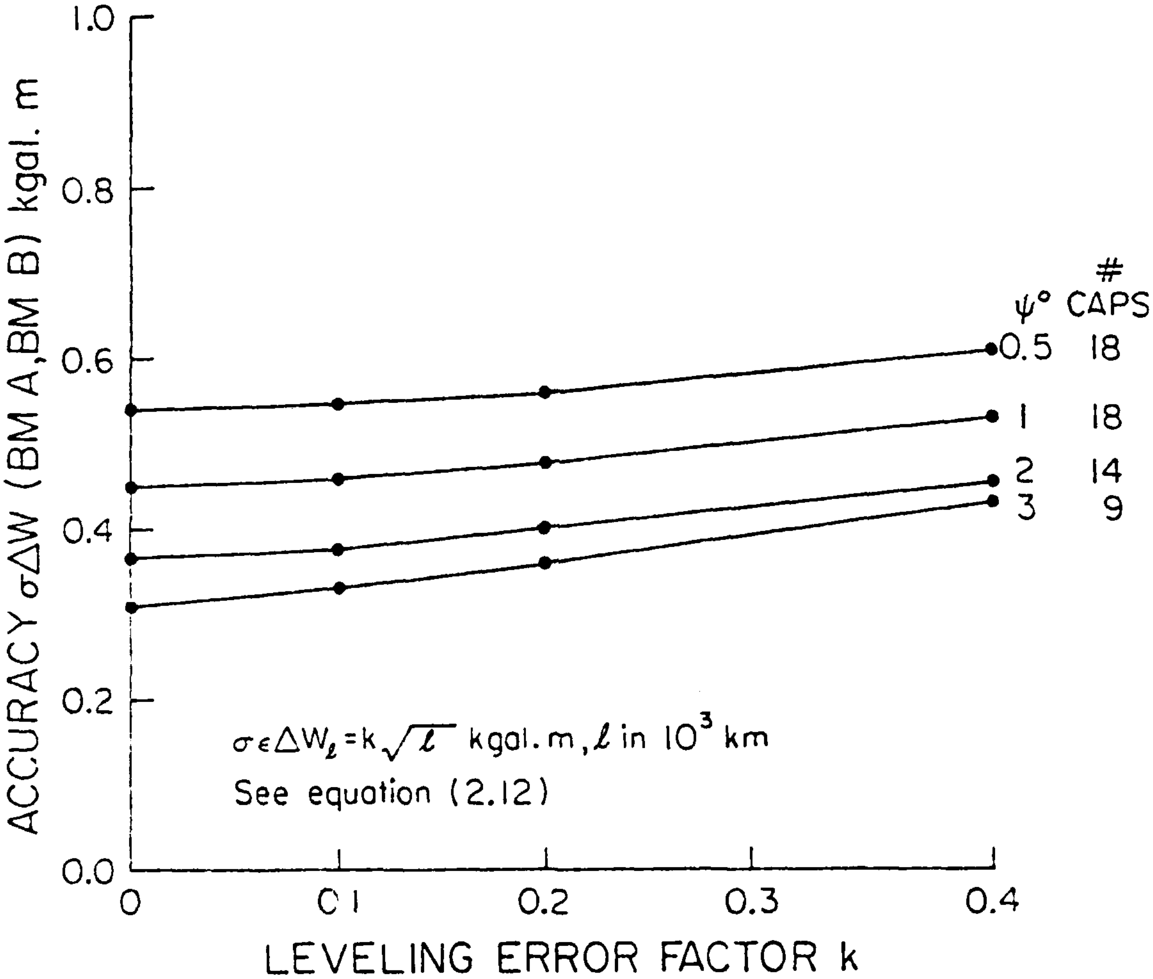
The effect of leveling errors on $\sigma\Delta W$(BMA,BMB) has been plotted in [illegible] 6.5, including some additional results for $\psi = 3°$. We recall [illegible] that a conservative estimate of potential differences [illegible] over intra-continental distances is given by $0.1\sqrt{\ell}$ [illegible] km (see (2.12)). Even if this estimate is doubled [illegible], the effect on $\sigma\Delta W$ is not large (see last 3 rows of Table [illegible]). This means that even very conservative estimates of leveling errors do not significantly affect the accuracy estimates of Europe - USA vertical datum connection.

Table 6.2 Variation in Accuray Estimate $\sigma\Delta W$(BMA,BMB) for Europe-USA Vertical Datum Connection due to: (a) s.d. (σs) of shift of origin of SLR stations from Geocenter; (b) s.d. ($\sigma\varepsilon r$) of radial position error in SLR stations; and (c) accuracy estimate ($\sigma\varepsilon\Delta W_\ell$) of potential difference by leveling between cap centers: $\sigma\varepsilon\Delta W_\ell = k(0.1\sqrt{\ell})$ kgal.m, ℓ in 10^3 km. Other particulars as in Table 6.1.

Capsize $\psi°$			2°	1°	0.5°
P_i: # Caps in Europe			4	4	4
Q_j: # Caps in USA			10	14	14
BM A			WETZEL	WETZEL	WETZEL
BM B			PLTTVX	PLTTVX	PLTTVX
Shift σs(cm)	Position $\sigma\varepsilon r$ (cm)	Leveling k	Accuracy Estimate $\sigma\Delta W$(BMA,BMB) kgal.m		
0	0	0	.37	.45	.54
10	0	0	.37	.46	.54
20	0	0	.39	.47	.55
30	0	0	.41	.49	.57
40	0	0	.45	.52	.60
50	0	0	.49	.56	.63
0	10	0	.37	.46	.54
0	20	0	.38	.46	.54
0	30	0	.39	.47	.55
0	40	0	.41	.48	.56
0	50	0	.43	.50	.57
0	0	1	.38	.46	.54
0	0	2	.40	.48	.56
0	0	4	.45	.53	.61

Figure 6.5: Accuracy Estimate $\sigma W(BMA,BMB)$ of Europe - USA Vertical Datum Connection. Variation Due to Leveling Errors.

No Geocentricity Error of Coordinate System.
No Station Position Error.
See Table 6.2.



6.3 Optimum Choice of Anomaly Cap Radius

We have already examined the effect of variation in anomaly data cap size ψ on $\sigma\Delta W(BMA,BMB)$ in Section 6.1, and it appears that $\psi = 1°$ or $\psi = 0.5°$ may be satisfactory considering the comparatively slow variation in $\sigma\Delta W$ due to other errors examined in Section 6.2. To determine an optimum ψ, we need to adopt conservative estimates of uncertainties of errors in Section 6.2; and consider if a lower value of ψ, requiring lesser field work of establishing $\Delta W_{\ell}(P_i,P')$ for modified gravity anomalies (see (4.9)*) in each cap, would not cause too large an increase in $\sigma\Delta W(BMA,BMB)$.

A comparison of the origin of SLR station coordinates in 1980 with the origin determined in a previous solution in 1979 gave a shift of about 15 cm (Smith et al., 1982). Hothem et al. (1982) estimated the relationship of co-located SLR station coordinates and the Doppler coordinate systems. After allowing for about 4 meter shift in the Z direction for the Doppler coordinate system, they estimated the difference in the origins of the two coordinate systems to be about 40 cm in X, 70 cm in Y and 10 cm in Z directions. Though the above comparison was strictly not with the SL4 solution (Smith et al., 1982), and the origin of the SL4 system may be located more precisely with the geocenter, we may assume a conservative estimate of σs in (2.11) as 30 cm.

The formal standard deviation of the annual mean SLR station heights in the SL4 system, based on the random error estimate of laser ranges, is of the order of 1 to 2cm (D. Christodoulidis, personal communication, July 1982). The standard deviation of the monthly solutions of SLR station heights from Lageos data in 1979-80 varies from 7 to 12 cm (Smith et al., 1982). A conservative estimate of radial errors in SL4 station position may be assumed as $\sigma\varepsilon r$ = 10 cm.

A conservative estimate cf errors in potential difference through leveling over intra-continental distance was discussed in (2.12) as $\sigma\varepsilon\Delta W_{\ell} = 0.1\sqrt{\ell}$ kgal.m, ℓ in 10^3 km. We give in Table 6.3 the accuracy estimate of Europe - USA vertical datum connection for $\sigma\varepsilon\Delta W_{\ell}$ as

$0.1\sqrt{\ell}$, also for $0.2\sqrt{\ell}$; and to emphasize that even very large leveling errors do not significantly affect $\sigma\Delta W(BMA,BMB)$, we also give results for $\sigma\epsilon\Delta W_\ell$ as $0.4\sqrt{\ell}$ kgal.m, ℓ in 10^3 km. Anomaly data in a uniform pattern around each cap center is considered for $\psi = 1°$ and $0.5°$; and also given for comparison for $\psi = 2°$ as well as $\psi = 0.25°$ and $\psi = 0.1°$. The anomaly density $\Delta\psi$ was kept at 10', as a higher density does not cause any significant improvement in $\sigma\epsilon T$ (Figure 4.2). However, to ensure that at least 3 ring averages are used for estimating $T(P_i)$ or $T(Q_j)$, the density was increased for $\psi = 0.25°$ and $\psi = 0.1°$. The number of modified anomalies in a uniform pattern, and $\sigma\epsilon T$ at each cap center are also given in Table 6.3.

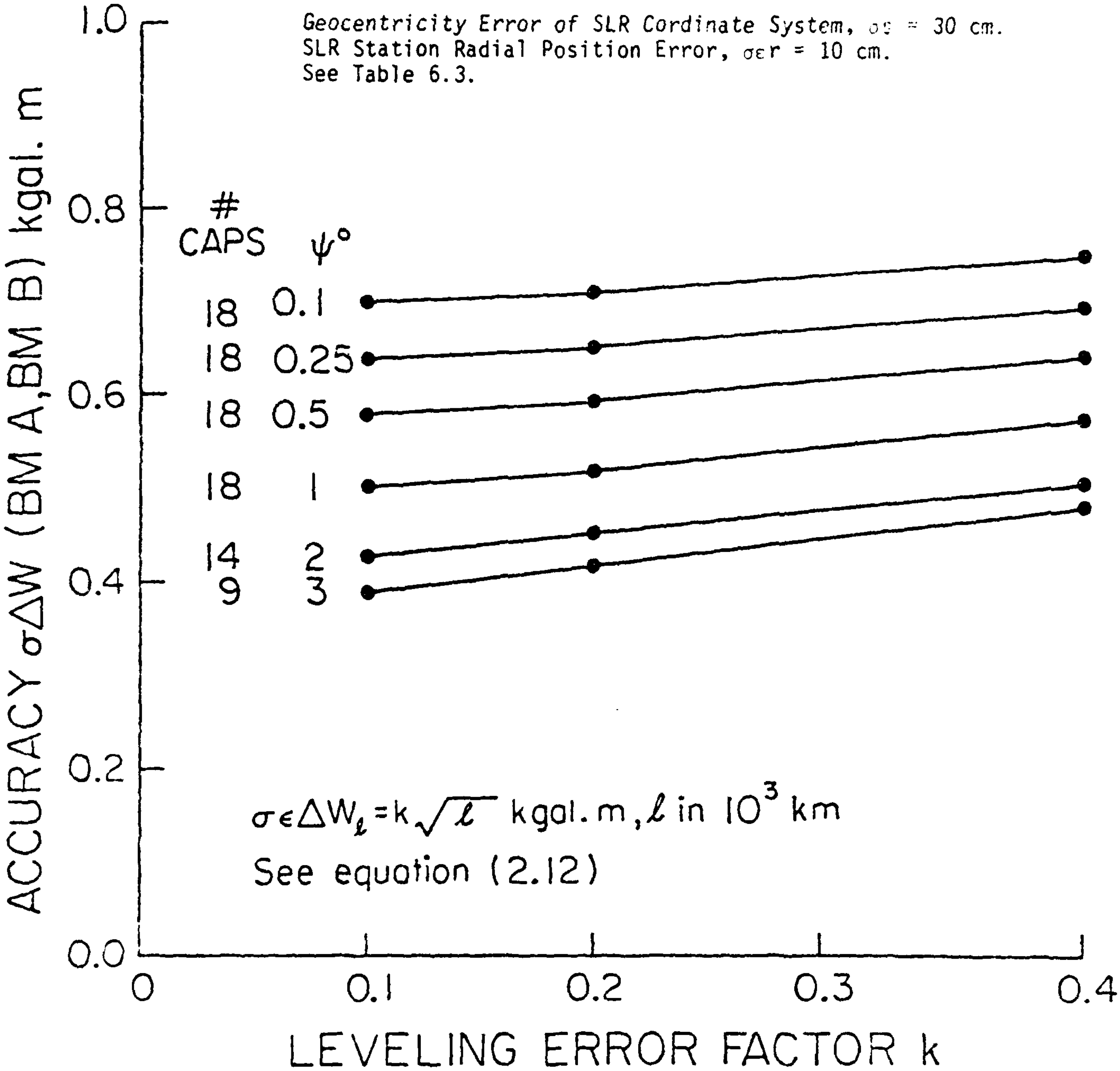
The accuracy estimates $\sigma\Delta W(BMA,BMB)$ in the last three rows of Table 6.3, and additional values for $\psi = 3°$, are shown in Figure 6.6. We note that a conservative accuracy estimate for Europe - USA vertical connection with anomaly data caps of spherical radius of 1° and 0.5° is 0.50 and 0.58 kgal.m, which changes to .52 and .59 kgal.m respectively. even if the estimate of leveling errors is doubled. The accuracy estimates will be further improved, when additional laser ranging stations in Europe in Spain, Switzerland and Greece are included in the Lageos SLR network.

Considering that the requirement of field work in each cap for providing leveling connections from cap center to the modified anomalies (see (4.9)*) is very considerably reduced for $\psi = 0.5°$ as compared to $\psi = 1°$ (by about one-third the number of anomalies over one-fourth the area), the optimum choice for anomaly data cap radius appears to be $\psi = 0.5°$. This leads to a very modest data requirement of about 50 to 60 anomalies at a rough spacing of 15 to 20 km around the SLR stations in a diameter of about 120 km. Such anomalies, with leveling connection to the cap center, are already available around most SLR stations, or may be established with modest field work. These anomalies may then be used to predict the modified anomalies in a uniform pattern, described below (4.7)*.

Table 6.3 Accuracy Estimate $\sigma\Delta W(BMA,BMB)$ for Europe-USA Vertical Datum Connection.
Optimum Choice of Anomaly Data Capsize $\psi°$.
Normal Gravitational Field, December 1981 Potential Coefficients.
s.d., σ_S, of shift of origin of SLR stations from geocenter
σ_S = 30 cm
s.d, $\sigma\hat{\varepsilon}r$, of radial position error in SLR stations: $\sigma\varepsilon r$ = 10 cm
s.d., $\sigma\varepsilon\Delta W_\ell$, of potential difference errors by leveling between cap centers: $\sigma\varepsilon\Delta W_\ell = k(0.1\sqrt{\ell})$ kgal.m, ℓ in 10^3 km, k = 1, 2, 4.
s.d, $\sigma\varepsilon\Delta g^*$ = 2 mgals.

Capsize $\psi°$	2°	1°	0.5°	0.25°	0.1°
P_i: # Caps in Europe	4	4	4	4	4
Q_j: # Caps in USA	10	14	14	14	14
BM A	WETZEL	WETZEL	WETZEL	WETZEL	WETZEL
BM B	PLTTVX	PLTTVX	PLTTVX	PLTTVX	PLTTVX
Anomaly density, $\Delta\psi$, in uniform pattern	10'	10'	10'	7.5'	3'
Approx. diameter of anomaly data cap (km)	440	220	110	55	22
# Anomalies per cap	475	139	43	19	19
# Anomalies ring averages per cap	13	7	4	3	3
Accuracy of predicted T in a cap ($\sigma\hat{\varepsilon}T$ - kgal.m)	.51	.67	.82	.95	1.06
Potential Difference Errors Through Leveling $\sigma\varepsilon\Delta W_\ell(BMA,P_i)$; $\sigma\varepsilon\Delta W_\ell(BMB,Q_j)$	$\sigma\Delta W(BMA,BMB)$ kgal.m				
$\sigma\varepsilon\Delta W_\ell = 0.1\sqrt{\ell}$ kgal.m, ℓ in 10^3km	.43	.50	.58	.64	.70
$= 0.2\sqrt{\ell}$	.45	.52	.59	.65	.71
$= 0.4\sqrt{\ell}$	.51	.57	.64	.70	.75

Figure 6.6: Accuracy Estimate σW(BMA,BMB) of Europe - USA Vertical Datum Connection. Optimum Choice of Anomaly Cap Radius $\psi°$.



7. ACCURACY ESTIMATE FOR OTHER VERTICAL DATUM CONNECTIONS

We apply in this section the conclusions of Section 6.3 to determine the accuracy estimate for other vertical datum connections. The datum connections considered were that of U.S. respectively with Australia, Hawaii, Peru (in South America), Bahamas and Bermuda (in mid-Atlantic off the U.S. East Coast). There are two SLR stations at Orroral and Yarragadee on the East and West coast of Australia, see Figure 3.1; and one SLR station each at other sites, assuming that a satisfactory intra-continental leveling connection may not be available between the two SLR station sites in South America at Arequipa, Peru and Natal, Brazil. In all cases, 14 SLR station sites were considered in U.S. First, the accuracy estimates $\sigma\Delta W(BMA,BMB)$ were obtained with anomaly data cap radius ψ around each SLR station as $0.5°$; and as $\sigma\Delta W(BMA,BMB)$ was comparatively larger than in Section 6 due to smaller number of caps in the vertical datum connection, $\sigma\Delta W(BMA,BMB)$ was also obtained with $\psi = 1°$. In both cases of $\psi = 0.5°$ and $\psi = 1°$, $\sigma\Delta W(BMA,BMB)$ was obtained with different variations σs, $\sigma\varepsilon r$, $\sigma\varepsilon\Delta W_\ell$ as in Section 6.2 (Table 6.2). We quote below in Table 7.1 only the results for the standard deviations σs, shift of SLR coordinate system from geocenter, as 30 cm; $\sigma\varepsilon r$, radial position error of SLR stations, as 10 cm; and $\sigma\varepsilon\Delta W_\ell$, potential difference between cap centers, as $0.1\sqrt{\ell}$ kgal.m, where ℓ is the length of level line in 10^3km. $\sigma\Delta W(BMA,BMB)$ is also given, when none of the above errors are considered, i.e., each of σs, $\sigma\varepsilon r$, $\sigma\varepsilon\Delta W_\ell$ set as zero. This allows for a ready comparison with results in Section 6, where we had 4 SLR sites in Europe, while we have now only 2 SLR sites for the vertical datum connection with Australia, and only 1 SLR site for other vertical datum connections. The accuracy estimate $\sigma\Delta W(BMA,BMB)$ were almost similar for the latter 4 datum connections, i.e. of U.S. respectively with Hawaii, Peru, Bahamas, Bermuda. Hence the results for only one of these, i.e. U.S. - Bermuda vertical datum connection have been given in Table 7.1.

Table 7.1 Accuracy Estimate of U.S. and Other Vertical Datum Connections.
σs = s.d. of shift of origin of SLR stations from geocenter; σs = 30 cm
$\sigma\delta r$ = s.d. of radial position errors in SLR stations; $\sigma\delta r$ = 10 cm
$\sigma\delta\Delta W_\ell$ = s.d. of potential difference errors by leveling between cap centers in a region
= $0.1\sqrt{\ell}$ kgal.m, ℓ in 10^3 km.
Other particulars as in Table 6.3

Vertical Datum Connections →	U.S.-Europe	U.S.-Australia	U.S.-Bermuda
No. of Caps in U.S.	14	14	14
No. of Caps in Other Datum	4	2	1*
Anomaly Data Capsize $\psi = 0.5^\circ$	Accuracy Estimate $\sigma\Delta W(BMA,BMB)$ kgal.m		
$\sigma s = \sigma\delta r = \sigma\delta\Delta W_\ell = 0$	.54	.65	.88
σs = 30 cm, $\sigma\delta r$ = 10 cm, and $\sigma\delta\Delta W_\ell = 0.1\sqrt{\ell}$ kgal.m, ℓ in 10^3 km	.58	.73	.89
Anomaly Data Capsize $\psi = 1^\circ$	Accuracy Estimate $\sigma\Delta W(BMA,BMB)$ kgal.m		
$\sigma s = \sigma\delta r = \sigma\delta\Delta W_\ell = 0$	.45	.53	.72
σs = 30 cm, $\sigma\delta r$ = 10 cm, and $\sigma\delta\Delta W_\ell = 0.1\sqrt{\ell}$ kgal.m, ℓ in 10^3 km	.50	.62	.73

* Similar results were obtained for other vertical datum connections, with only one SLR station in the second region, i.e. U.S. and respectively Bahamas, Hawaii, Peru.

We note from Table 7.1 that the accuracy estimate of the vertical datum connection is poor (.73 kgal.m) even with $\psi = 1°$, when we have only one SLR station in the second region. When the number of SLR stations in the second region increases to 2 and 4 respectively, the accuracy of the vertical datum connection is 0.62 kgal.m with $\psi = 1°$, and 0.58 kgal.m with $\psi = 0.5°$ respectively. The number of stations available for the vertical datum connection has a greater effect for improving the accuracy of datum connection, as compared to the anomaly data cap size $\psi°$.

To confirm the above conclusion, we show in Table 7.2 the accuracy estimate of U.S.-Europe vertical datum connection with $\psi = 0.5°$ for three cases, when the number of stations in Europe are taken as 2,4 and 7 respectively. For the first case, we consider only two SLR stations in Europe at Wetzell, Germany and Grasse, France (see note below Table 6.1 and Figure 6.1). For the second case, we consider 4 stations in Europe as in Section 6. For the third case, we consider three additional stations in Europe at Dionysos, Greece; Zimmerwald, Switzerland; and San Fernando, Spain. The coordinates of these additional stations are not currently available in the SL4 (Smith et al., 1982) solution of Lageos SLR network, but these coordinates are likely to be available in subsequent solutions.

Table 7.2 Accuracy Estimate of U.S.-Europe Vertical Datum Connection. Variation Due to Number of SLR stations in Europe.
Anomaly Data Capsize $\psi = 0.5°$
σs = 30cm, σr = 10 cm, $\sigma \Delta W_o = 0.1 \sqrt{\ell}$ kgal.m
(see explanation in Table 7.1)
Other particulars as in Table 6.3.

No. of Caps in U.S.	14	14	14
No. of Caps in Europe	7	4	2
Anomaly Data Capsize $\psi = 0.5°$	Accuracy Estimate $\sigma \Delta W$(BMA,BMB) kgal.m		
σs = 30 cm, σr = 10 cm, and $\sigma \Delta W_o = 0.1 \sqrt{\ell}$ kgal.m, ℓ in 10^3 km	.49	.58	.74

A comparison of the last row and the first columns of Tables 7.1 and 7.2 shows that the accuracy estimate of U.S.-Europe vertical datum is about the same when we use $\psi = 0.5°$ but have three additional stations in Europe as compared to $\psi = 1°$ but without the three additional stations. However, there is a large increase in field work of leveling from the cap centers to the anomalies in the latter case of $\psi = 1°$. Hence, the number of stations available for the vertical datum connection has a greater effect for improving the accuracy of datum connection, as compared to the anomaly data capsize $\psi°$.

8. SUMMARY AND CONCLUSIONS

We have investigated the procedures recommended by Colombo (1980a) for establishing a 'world vertical network', and have applied the results to determine the current accuracy estimate, in particular, of the Western Europe-USA vertical datum connection.

The principle of the method is to determine the potential difference between two 'bench-marks' BM A and BM B in the two regions A and B, without a precise knowledge of the mean sea level, i.e. without a precise knowledge of the 'sea surface topography' or the geoid, in particular the zero degree term in the earth's gravitational potential, the effect of which nearly cancels out in determining the potential difference. The estimate of the potential difference $\Delta W(BMA,BMB)$ is more precisely obtained through usual least squares technique by redundant observations, when we consider the potential difference between several pairs of benchmarks P_i and Q_j, respectively in the two regions A and B, and relate the potential difference of each pair to $\Delta W(BMA,BMB)$ by determining $\Delta W(BMA, P_i)$ and $\Delta W(BMB, Q_j)$ through classical leveling and gravity observations.

Hence, the requirements for establishing a vertical datum connection are:

(a) Precise station positions, particularly in the radial component, in an earth-centered coordinate system, to determine the normal gravitational potential $U(P_i)$ or $U(Q_j)$ and the centrifugal potential $\Phi(P_i)$ or $\Phi(Q_j)$ -- see (2.1). The possible shift of the origin of coordinate system from the geocenter is considered through correlated errors in the normal gravitational potential estimates of the stations (see (2.10)). The random errors in station positions in the established coordinate system are considered through (2.7) and (2.8). Any systematic effect of a scale factor, the radial component of all station positions being larger or smaller, is nearly cancelled out in determining potential differences.

(b) An accurate description of the earth's gravitational field in terms of potential coefficients in a spherical harmonic expansion to a high degree N. This enables computation of U through (2.6). The errors in the potential coefficients are considered in defining the anomalous potential

T in (3.2) through the estimated standard deviations of the potential coefficients in (3.4) to (3.7). The potential coefficients for degrees higher than N are modelled through anomaly degree variance models -- see (3.8) to (3.11).

(c) Gravity anomalies, residual to degree N, at an optimum density and extent, or capsize, around each station to predict $T(P_i)$ or $T(Q_j)$ through (4.1). The available anomalies are first used to predict a set of anomalies in a uniform pattern (see below (4.7)*) for an efficient algorithm utilizing 'ring averages' (4.4), which greatly reduces the size of matrices in the collocation formulas (4.1). The choice of optimum density and extent of the anomalies in the pattern are important for a precise estimate of T (Section 4.3). The propagation of random errors in the anomalies to accuracy estimate of T is considered in Section 4.5. A systematic error in gravity anomalies is nearly cancelled in the potential differences as the same density, extent, and pattern of anomalies is used around each station.

(d) Estimate of potential differences through leveling between the locations of gravity anomalies in the 'cap' and the station or cap center. This enables the computation of 'modified' gravity anomalies (Section 4.2), such that the modified anomalies are referenced to the normal potential of the cap center instead of being referenced to the potential of the geoid, see (4.8)*, (4.9)*. The precise knowledge of the geoid is therefore not required. In subpara. (c) above, modified gravity anomalies are used for predicting T, but as the covariances in the collocation formula are computed utilizing anomaly degree variance model, the accuracy estimate of T are corrected as in (4.13).

(e) Estimate of potential differences $\Delta W_\ell(BMA,P_i)$ and $\Delta W_\ell(BMB,Q_j)$, through leveling and gravity observations, over intra-continental distances in each region A and B. The effect of random errors in ΔW_ℓ are considered in Section 2.3. Any systematic error in ΔW_ℓ, which is dependent on the routes used from (BMA,P_i) or (BMB,Q_j), will become apparent when equations (2.1) or (2.3) are formed.

(f) An adequate number of stations for establishing the vertical connection to enable a more precise estimate of $\Delta W(BMA,BMB)$. The number of equations (2.1) or (2.3) are one less than the total number of stations in both regions A and B.

8.1 Recommendations of Present Study

The above requirements are already fulfilled to a sufficient extent to attempt the vertical datum connection between Western Europe and USA. A modest amount of field work may be necessary before actual computations are done. We first list the current possibilities, and identify the errors whose effects on the vertical datum connection was investigated in this study; and also briefly mention the highly near-future improvements:

(a) If we count the repeated occupations of adjacent marks in a vicinity as one station, i.e. using only one of the marks, we have 29 station positions (Figure 3.1) in the Lageos Satellite ranging network in the SL4 coordinate system (Smith et al., 1982). We used a very conservative estimate of the standard deviation of the shift of SL4 coordinate system from the geocenter as 30 cm, and we used the standard deviation of the radial component in the station positions in the SL4 coordinate system as 10 cm (see Section 6.3). These are likely to be substantially improved in the subsequent SL5 solution. However, this is not a critical factor and may improve the accuracy estimate of the vertical connection by only a marginal amount, say .03 kgal.m (see Table 6.2).

(b) We used the earth's gravitational field model developed by Rapp (1981), which was termed in this study as December 1981 potential coefficients field, and was complete up to degree 180. The standard deviations of the potential coefficients are perhaps slightly pessimistic (Lachapelle and Rapp, 1982, p. 4). Any improvement in the accuracy of the gravitational field will result in significant improvement of the accuracy estimate of the vertical connection -- see Figure 4.3 and Table 4.3. A revised estimate of the gravity field, with accurately determined low degree (say $N \leq 10$) potential coefficients, may improve the accuracy estimate of the vertical connection by about 0.1 kgal.m.

(c) The optimum density of gravity anomalies is at spacing of about 10', roughly 15 to 20 km apart -- see Figure 4.2 and Table 4.2. The optimum extent of anomalies, or the radius ψ of the anomaly data cap around each station, is obtained by a judicious balance of the practical feasibility of the increased field effort with larger ψ versus the lowered precision of the accuracy estimate of the vertical connection with smaller ψ. An important conclusion is that, unlike Colombo's original proposal of $\psi = 5°$, ψ may certainly be reduced to 1° (Sections 6.1 and 6.2) without significant deterioration in vertical connection accuracy, particularly because more stations are available with reduced cap size (compare Figures 6.2, 6.3 and 6.4). A further reduction in ψ to 0.5° is discussed below. A very encouraging conclusion is the lack of sensitivity of the vertical connection accuracy to the random errors in gravity anomalies (Figure 4.5 and Table 4.4). With several SLR stations located in the coastal areas, and anomaly data cap covering marine areas, it is very helpful to know that even large random errors of 10 mgals (1σ) in anomalies will lower the accuracy of vertical connection only by about 0.01 kgal.m.

(d) *Gravity observations at approximate spacing of 15 to 20 km may be* available around most stations, but leveling observations may be required to determine potential differences from the station at cap center to the locations of anomalies. When we consider that in case of $\psi = 0.5°$, about one-third anomalies are required to be connected by leveling over one-fourth the area around each station (Table 6.3), as compared to the case of $\psi = 1°$, while the accuracy of vertical connection is lowered by only about 0.07 kgal.m. it appears that the optimum extent of gravity anomalies is a circle of diameter of about 120 km corresponding to $\psi = 0.5°$.

(e) The vertical connection accuracy is quite insensitive to even very conservative estimates of random errors in leveling over intra-continental distances, see Figures 6.5 and 6.6 and equation (2.12). The currently available estimates of $\Delta W_{\ell}(BMA, P_i)$ and $\Delta W_{\ell}(BMB, Q_j)$ are therefore adequate for the vertical datum connections. This is a very encouraging conclusion in view of concern over accuracy of leveling over intra-continental distances, e.g. Kumar and Soler (1981). Further, the effect of systematic errors in [illegible] should become apparent by the examination of residuals in equations (2.1).

(f) The vertical connection accuracy improves sharply with the increase in the total number of stations, see the last two rows of Table 6.1. We have presently used 4 stations in Europe and 14 stations in USA. This should be increased in the very near future by another three stations when the coordinates of laser ranging stations in Spain, Switzerland and Greece are also solved for in the Lageos network. The number of stations would increase further in the next couple of years with the utilization of Transportable laser ranging stations (NASA, 1982).

The present accuracy estimate, with the data used in this study (see (a) to (f) above), for the vertical datum connection between Western Europe and USA is about 60 kgal.cm, see Table 6.3. The improvement in this estimate would come primarily from increased number of stations and more accurate estimates of earth's gravity field, subparas (f) and (b) above. The very near future prospects for the vertical datum connection accuracy are likely to be about 50 kgal.cm. This is about one-third of the classical vertical connection accuracy through tide gage determinations due to sea surface topography.

The various requirements detailed in Section 8 are already available, or easily accessible. The main additional field effort is the estimate of potential differences at each SLR station by establishing leveling connections from the station to the locations of about 50 gravity anomalies at a spacing of 15-20 km in a circle of diameter of about 120 km around the station.

8.2 Recommendations for Further Studies

The following topics were not covered in the present investigation, and should be studied further:

(a) An optimum algorithm for predicting anomalies in a uniform pattern, see below (4.7)*, around each SLR station from the nearest available anomalies.

(b) An optimum procedure for computing modified gravity anomalies from the available gravity observations; and consideration of the corrections to be applied to such observations.

(c) The problem of determining the absolute value of geopotential at BM A or BM B, with better or comparable accuracy to the potential difference W(BMA,BMB) of the vertical datum connection.

(d) The possibility and the accuracy of sea surface topography estimates by connecting the coastal SLR stations to tide gages and comparing with the mean sea level determinations at those tide gages.

A better insight into the establishment of the 'World Vertical Network' will be obtained by initiating the computations of the vertical datum connection between Western Europe and USA. Most of the data is easily accessible, and a very modest field effort is needed to obtain the remaining data. A very intriguing question is the possibility of revealing any systematic errors in leveling observations over intra-continental distances by examining the residuals of equations (2.1). This may be the appropriate time to start such investigations in view of the proposed redefinition of the vertical networks in North America.

References

Colombo, O.L., Optimal Estimation from Data Regularly Sampled on a Sphere with Applications in Geodesy, Report No. 291, Department of Geodetic Science, The Ohio State University, Columbus, September 1979.

Colombo, O.L., A World Vertical Network, Report No. 296, Department of Geodetic Science, The Ohio State University, Columbus, February 1980 a.

Colombo, O.L., Transoceanic Vertical Datum Connections, Proceedings Second International Symposium on Problems Related to the Redefinition of North American Vertical Geodetic Networks, Ottawa, May 1980 b, pp. 87-104.

Federal Geodetic Control Committee, Specifications to Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys, National Ocean Survey, NOAA, Rockville, Maryland, June 1980.

Heiskanen, W.A. and H. Moritz, Physical Geodesy, W.H. Freeman and Company, San Francisco, 1967.

Jekeli, C., An Investigation of Two Models for the Degree Variance of Global Covariance Functions, Report No. 275, Department of Geodetic Science, The Ohio State University, Columbus, September 1978.

Hothem, L.D., T. Vincenty and R.E. Moose, Relationship Between Doppler and Other Advanced Geodetic System Measurements Based on Global Data, Paper presented at the Third International Geodetic Symposium on Satellite Doppler Positioning, Las Cruces, February 1982.

Kumar, M. and T. Soler, Southern California Releveling and Its Implications With Respect to the Sea Slope Problem, Marine Geodesy, Vol. 5, No. 3, 1981, pp. 237-264.

Lachapelle, G. and R. Gareau, Status of Vertical Geodetic Networks in Canada, Proceedings Second International Symposium on Problems Related to the Redefinition of North American Vertical Geodetic Networks, Ottawa, May 1980, pp. 49-71.

Lachapelle, G. and R.H. Rapp, Estimating of Disturbing Potential Components with Emphasis on North America, Paper presented at the CIS Centennial Convention, The Canadian Institute of Surveying, Ottawa, 1982.

Lennon, G.W., Mean Sea Level as a Reference for Geodetic Leveling, The Canadian Surveyor, Vol. 28, No. 5, December 1974, pp. 524-530.

Mather, R.S., The Role of the Geoid in Four-Dimensional Geodesy, Marine Geodesy, Vol. 1 No. 3, 1978, pp. 217-252.

Moritz, H., Advanced Physical Geodesy, Abacus Press, Kent, 1980.

NASA, NASA Geodynamics Program: Annual Report for 1980, NASA Technical Memorandum 84010, Washington, D.C. October 1981.

NASA, Crustal Dynamics Project Bulletin, Vol. 1 No. 1, Goddard Space Flight Center, Greenbelt, Maryland, April 1982.

NASA, NASA Geodynamics Program: Annual Report for 1981, NASA Technical Memorandum 85126, Washington D.C., August 1982.

NOAA, Land Gravity Data for Conterminous United States, Data Announcement 82-TGB-02, National Geophysical and Solar-Terrestrial Data Center, Boulder, Colorado, 1982.

Rapp, R.H., A Global 1°x 1° Anomaly Field Combining Satellite, Geos-3 Altimeter and Terrestrial Anomaly Data, Report No. 278, Department of Geodetic Science, The Ohio State University, Columbus, September 1978.

Rapp, R.H., Potential Coefficient and Anomaly Degree Variance Modelling Revisited, Report No. 293, Department of Geodetic Science, Ohio State University, Columbus, September 1979.

Rapp, R.H., Precise Definition of the Geoid and Its Realization for Vertical Datum Applications, Proceedings Second International Symposium on Problems Related to the Redefinition of North American Vertical Geodetic Networks, Ottawa, May 1980, pp. 73-86.

Rapp, R.H., The Earth's Gravity Field to Degree and Order 180 Using Seasat Altimeter Data, Terrestrial Gravity Data, and Other Data, Report No. 322, Department of Geodetic Science, Ohio State University, Columbus, December 1981.

Smith, D.E., P.J. Dunn, D. Christodoulidis and M. Torrence, Global Baselines from Laser Ranging, Fourth Conference on NASA Geodynamics Program, Goddard Space Flight Center, Greenbelt, Maryland, January 1982.

Vanicek, P., R.O. Castle and E.I. Balazs, Geodetic Leveling and Its Applications, Reviews of Geophysics and Space Physics, Vol. 18 No. 2, May 1980, pp. 505-524.